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Jordan–Hölder, modularity and distributivity in non-commutative algebra

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Abstract

A study of lattices of subgroups or subrings adequate for non-commutative homological algebra can be pursued in a setting of *weakly exact* categories, which extend the Puppe-exact ones [D. Puppe, Korrespondenzen in abelschen Kategorien, Math. Ann. 148 (1962) 1–30; B. Mitchell, Theory of Categories, Academic Press, New York, 1965; P. Freyd, A. Scedrov, Categories, Allegories, North-Holland Publishing Co., Amsterdam, 1990] and the semi-abelian ones [G. Janelidze, L. Márki, W. Tholen, Semi-abelian categories, J. Pure Appl. Algebra 168 (2002) 367–386; F. Borceux, A survey of semi-abelian categories, in: Galois Theory, Hopf Algebras, and Semiabelian Categories, in: Fields Inst. Commun., vol. 43, Amer. Math. Soc., Providence, RI, 2004, pp. 27–60; F. Borceux, D. Bourn, Mal'cev, Protomodular, homological and semi-abelian categories, in: Mathematics and its Applications, vol. 566, Kluwer Academic Publishers, Dordrecht, 2004], and are essentially based on a notion of γ-category introduced by Burgin [M.S. Burgin, Categories with involution and correspondences in γ-categories, Tr. Mosk. Mat. Obs. 22 (1970) 161–228; Trans. Moscow Math. Soc. 22 (1970) 181–257]. In this context, subobjects form *w-modular w-lattices*, equipped with a normality relation. The free w-modular w-lattice generated by two chains with normality conditions is determined and proved to be *weakly distributive*, by a construction inspired by the well-known Birkhoff theorem for free modular lattices [G. Birkhoff, Lattice Theory, 3rd ed., in: Amer. Math. Soc. Coll. Publ., vol. 25, 1973]. We show that this theorem is relevant for the study of double filtrations, much in the same way as the Birkhoff theorem in the commutative case; similarly, it should be of use in the study of spectral sequences.

MSC: 18G50; 06C05; 06DXX; 18E10

0. Introduction

A relevant part of homological algebra, from the homology of chain complexes to spectral sequences, is based on the study of *subquotients* and induced morphisms between them (a subquotient of an object A is a quotient H/K of a subobject $H \subset A$). Relevant tools for this study are:

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(a) subobjects, their direct and inverse images,

(b) categories of relations.

The second point is already stressed in Mac Lane's text on Homology [18]. But here we will rather focus on the first.

In *commutative* homological algebra, a suitable setting for dealing with lattices of subobjects is given by Puppe-exact categories (cf. 1.1), rather than the classical, stronger notion of abelian category. In fact, let us start from an abelian category $\mathbf{E}$ (or even from the category Ab of abelian groups). Then, every object A has a *modular lattice* $\mathrm{Sub}_{\mathbf{E}}(A)$ of subobjects, and every morphism $f: A \to B$ produces, by direct and inverse images, a "covariant Galois connection", i.e. an adjunction between the modular lattices $X = \mathrm{Sub}_{\mathbf{E}}(A)$, $Y = \mathrm{Sub}_{\mathbf{E}}(B)$

$$f_* : X \rightleftarrows Y : f^*, \qquad f^* f_*(x) \geq x, \qquad f_* f^*(y) \leq y \qquad (x \in X,\ y \in Y). \tag{1}$$

Such an adjunction satisfies a stronger property: it is a *modular connection* [13], i.e. a pair of increasing mappings such that (as is obvious in Ab):

$$f^* f_*(x) = x \vee f^*(0), \qquad f_* f^*(y) = y \wedge f_*(1) \qquad (x \in X,\ y \in Y). \tag{2}$$

Now, the category Mlc of all *modular lattices and modular connections* is Puppe-exact (see 1.1), but has no products nor sums, and the setting of abelian categories appears to be too restricted. But all we have said holds true in the wider context of Puppe-exact categories, and for any such $\mathbf{E}$ there is a functor *of subobjects* as above

$$\mathrm{Sub}_{\mathbf{E}} : \mathbf{E} \to \mathsf{Mlc}, \qquad A \mapsto \mathrm{Sub}_{\mathbf{E}}(A), \qquad f \mapsto (f_*, f^*), \tag{3}$$

which is exact in the ordinary sense of homological algebra – preserves the zero object, kernels and cokernels – and also reflects exactness (all this can be found in [13]).

Deeper motivations for not assuming in our setting the existence of finite products (or sums) are linked with a notion of *distributive homological algebra* developed in [14]: essentially, the main "theories" producing spectral sequences, from the filtered complex to the double complex or the exact couple, are *distributive*, in the sense that their classifying Puppe-exact category is so: it has distributive lattices of subobjects. But this property, which has relevant consequences, cannot even be formulated in the abelian context, since a distributive abelian category is necessarily trivial (equivalent to $\mathbf{1}$).

Now, moving to *non-commutative* homological algebra, the previous setting is no longer adequate: the category Gp of groups is not Puppe-exact, because it has non-normal subobjects. We shall use a wider notion of *weakly exact* category (1.2), essentially based on Burgin's γ-categories [9], which allows us to set up a similar treatment of subobjects. If $\mathbf{E}$ is weakly exact, each poset of subobjects $\mathrm{Sub}_{\mathbf{E}}(A)$ comes equipped with a normality relation $\lhd$, and has a structure of *w-modular w-lattice* (defined in 2.1 and 2.2); again, a morphism $f: A \to B$ produces, by direct and inverse images, a "covariant Galois connection" (f_*, f^*) whose components preserve order and normality, and satisfy the conditions (2); the category wMlc of w-modular w-lattices and such *wm-connections* will be proved to be weakly exact (Theorem 2.6) and to receive a *functor of subobjects* $\mathrm{Sub}_{\mathbf{E}} : \mathbf{E} \to \mathsf{wMlc}$ from every w-exact category; again, this functor is "exact" in the appropriate sense and reflects exactness (Theorem 2.7).

As a test of these notions and a basis for future developments, we prove the analogue of the Birkhoff theorem for the free modular lattice generated by two chains [1], which was a crucial point for the construction of Zeeman diagrams of spectral sequences [21,16] and, more formally, of the classifying Puppe-exact category of a filtered chain complex ([14], Part III): our Main Theorem here (4.1) shows that the free w-modular w-lattice generated by two (finite) chains with normality conditions is *weakly distributive*, and – again – can be realised within the lattice of parts of a (finite) rectangle of $\mathbb{N} \times \mathbb{N}$. We use this description as a guideline in the proof of a Jordan–Hölder theorem for w-exact categories (Theorem 4.4). Thus, the importance of distributivity also appears here, if in a weak form linked with the normality relation; and it should be possible to set up a study of non-commutative homological algebra by *classifying w-exact categories*, corresponding to the one developed in [14] for the commutative case, and single-out a notion of *w-distributive theory* where canonical isomorphisms are "coherent" (form transitive systems). Again, such developments would not be possible in a setting based on products, since the latter are in contradiction with distributivity (cf. 4.5).

Finally, even if we are not going to use here the categories of relations, let us recall that such constructions on Puppe-exact categories and their generalisations do exist, but are based on a "four-map factorisation" (as opposed to the ordinary "two-map factorisation" which one can use in the presence of products): see Calenko [10,11], Brinkmann–Puppe [8], and Burgin [9].

Outline. Section 1 reviews the setting of w-exact categories, in comparison with other settings for commutative and non-commutative algebra: Puppe-exact, Barr-exact and semi-abelian categories: see Proposition 1.5 and a comparative diagram in 1.6. Section 2 studies w-modular w-lattices, their homomorphisms and their adjunctions, and constructs the functor of subobjects of a w-exact category (2.7). Then, after a brief Section 3 on subquotients and induced morphisms, Section 4 contains our Main Theorem 4.1 and its applications; its proof is in Section 5 while the last section contains a few diagrammatic lemmas on w-exact categories used in the previous part.

1. Weakly exact categories and other settings

This section is a brief review of the main setting used here, and its relations with other contexts.

1.1. Puppe-exact categories

As discussed in the Introduction, this self-dual setting is relevant for studying commutative homological algebra.

A *Puppe-exact* category [20,19,12], *p-exact* for short, is a category with zero object where every map factors as a conormal epi followed by a normal mono. As a consequence, kernels and cokernels exist, all monos are normal, all epis are conormal; moreover, monos can be pulled-back along arbitrary arrows while epis can be pushed-out. However, products, general pullbacks, sums and general pushouts need not exist: it is well known that a p-exact category is abelian (with a well-determined additive structure) if and only if it has finite products, if and only if it has finite sums.

In an abelian category, the lattices of subobjects are modular; the same holds for all p-exact categories. More formally, one can construct (as indicated in the Introduction) a p-exact category Mlc of *modular lattices and modular connections*, so that every p-exact category $\mathbf{E}$ has a *functor of subobjects* $\mathrm{Sub}\colon \mathbf{E} \to \mathsf{Mlc}$ [13]; now, Mlc lacks products and sums.

As another example of a non-abelian p-exact category, consider the category $K\!-\!\mathsf{Prj}$ of projective spaces over a commutative field (cf. [13]). Or also the category $\mathcal{I}$ of sets and partial bijections (between subsets of domain and codomain), which has distributive lattices of subobjects and is the basis for constructing universal models of spectral sequences [14]. Our interest in *not assuming* the existence of products and sums rests also in admitting the *possibility of distributive lattices of subobjects*. (Note that, should this happen in an *abelian* category, the latter would be trivial, since the existence of a non-zero object A produces three subobjects in $A \oplus A$ which violate distributivity: the summands and the diagonal).

An equivalent, more explicit description of p-exactness can be obtained as follows. Let us start considering a category $\mathbf{E}$ with zero object, kernels and cokernels; then, every map f has a unique *normal factorisation* $f = mgp$, through the *normal coimage* $p = \mathrm{cok}\,(\mathrm{ker}\, f)$ and the *normal image* $m = \mathrm{ker}(\mathrm{cok}\, f)$

$$\begin{tikzcd}
\mathrm{Ker}\, f \arrow[r, tail, "\mathrm{ker}\, f"] & A \arrow[rr, "f"] \arrow[d, two heads, "p"'] & & B \arrow[r, two heads, "\mathrm{cok}\, f"] & \mathrm{Cok}\, f \\
& \mathrm{Cok}\,(\mathrm{ker}\, f) \arrow[rr, "g"] & & \mathrm{Ker}\,(\mathrm{cok}\, f) \arrow[u, tail, "m"'] &
\end{tikzcd} \tag{4}$$

Now, $\mathbf{E}$ is p-exact if and only if, for every f, this map g is an isomorphism.

Of course, the category Gp of all groups is not p-exact, since it has non-normal subobjects. Various settings, generally not selfdual, have been proposed for non-commutative homological algebra. We are mostly interested in the following one (essentially due to Burgin [9]), which generalises p-exact categories and – again – does not require products and sums, allowing thus for a weak form of distributivity and "set-based" classifying categories.

1.2. Weakly exact categories

A *weakly exact* category, or *w-exact* category, satisfies the following axioms:

(WE.1) there is a zero object 0; (a zero morphism will be written as $0: A \to B$ or 0_{AB});

(WE.2) every morphism u has a canonical factorisation $u = mp$, where p is a conormal epi (a cokernel of some map) and m is mono (such a factorisation is necessarily unique up to isomorphism; our category will always be provided with this canonical factorisation system; the terms "quotient" and "subobject", as well as the arrows "$\twoheadrightarrow$" and "$\rightarrowtail$", will always be used for conormal epis and arbitrary monomorphisms, respectively);

(WE.3) the pullback of a mono and an arbitrary morphism exists;

(WE.4) the pullback of a conormal epi along a mono is a conormal epi;

(WE.5) (Restricted Short Five Lemma) if, in the following commutative diagram

$$
\begin{tikzcd}
\bullet \arrow[r, tail, "k"] \arrow[d, equal] & \bullet \arrow[r, two heads, "p"] \arrow[d, tail, "f"'] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, tail, "k'"'] & \bullet \arrow[r, two heads, "p'"'] & \bullet
\end{tikzcd}
\tag{5}
$$

the rows are short exact (i.e. $k = \ker p$, $k' = \ker p'$) and f is a monomorphism, then the latter is an isomorphism;

(WE.6) the image of a normal mono by a conormal epi is a normal mono.

We say that the category $\mathbf{E}$ is *w*-exact* if its opposite category is w-exact; then, the canonical factorisation in $\mathbf{E}$ is by epimorphisms and normal monomorphisms.

When studying the posets of subobjects $\mathrm{Sub}_{\mathbf{E}}(A)$, in Section 2, it will be relevant to assume that $\mathbf{E}$ is well-powered. The axioms (WE.1–6) considered above, plus well-poweredness, are directly equivalent to Burgin's axioms for *γ-categories* [9].

Remarks 1.3. (a) In a w-exact category, every morphism $f: A \to B$ has a kernel $\ker f: \mathrm{Ker}\, f \rightarrowtail A$ (by (WE.1, 3)), as well as an image $\mathrm{im}\, f: \mathrm{Im} f \rightarrowtail B$, derived from the canonical factorisation. Because of the latter, f is mono if and only if $\ker f = 0$, while f is a conormal epi if and only if $\mathrm{im}\, f = 1_B$.

(b) A w-exact category is p-exact if and only if all its monomorphisms are normal (in which case all axioms (WE.3–6) are redundant), if and only if it is also w*-exact. The category of groups Gp and the category of rings Rng (without unit assumption) are just w-exact, while the category of pointed sets Set_* is just w*-exact.

(c) Here, we shall not develop the theory of w-exact categories; the reader is referred to Burgin's paper [9,15]. A few diagrammatic lemmas which we need are deferred to Section 6, including the (non-restricted) Short Five Lemma and characterisations of "mixed pullbacks" and pushouts of conormal epis.

1.4. Semi-abelian categories

We also want to compare w-exact categories with a more recent setting for non-commutative algebra, introduced in [17] and studied in various papers; see [5,6] for preliminary papers setting the bases of the theory, [3] for a survey, and the recent book [4] for a comprehensive study. Being based on the existence of finite limits and colimits, this setting can deal with subjects like the theory of semi-direct products [7], commutators, etc.

The axioms of a *semi-abelian category* can be written as follows:

(SA.1) there is a zero object 0;

(SA.2) pullbacks exist;

(SA.3) every morphism f factors as $f = mp$, where p is a coequalizer and m a monomorphism;

(SA.4) the pullback of a coequalizer along any morphism is again a coequalizer;

(SA.5) (Split Short Five Lemma) If in the following diagram

$$\begin{tikzcd} \bullet \arrow[r, "k"] \arrow[d, equal] & \bullet \arrow[r, shift left, "p"] \arrow[d, "f"'] & \bullet \arrow[l, shift left, "s"] \arrow[d, equal] \\ \bullet \arrow[r, "k'"] & \bullet \arrow[r, shift left, "p'"] & \bullet \arrow[l, shift left, "s'"] \end{tikzcd} \tag{6}$$

the squares commute, $k = \ker p$, $k' = \ker p'$, $ps = 1$ and $p's' = 1$ then f is iso;

(SA.6) every equivalence relation is effective (a kernel pair of a morphism);

(SA.7) the sum of two objects exists.

The axioms (SA.1–4) amount to saying that our category is regular with a zero object; adding (SA.6) means that it is also Barr-exact. The axiom (SA.5) amounts to Bourn protomodularity [5,6].

Proposition 1.5. *Every semi-abelian category is w-exact. More precisely:*

(a) *a finitely complete category with zero object is equivalently described by axioms* (SA.1–2)*, or by* (WE.1–2) *plus the existence of finite products;*

(b) *the axioms* (SA1–5) *amount to a w-exact category with finite products (or finite limits);*

(c) *a semi-abelian category amounts to a w-exact category with finite products and sums (or finite limits and colimits), where every equivalence relation is effective.*

Proof. (a) It is well known that a category is finitely complete if (and only if) it has a terminal object and pullbacks, if it has finite products and pullbacks of monos.

(b) First, let us take a w-exact category with finite products. For (SA.3–4), note that a cokernel is a coequalizer (trivially), and conversely any coequalizer p is a conormal epi: in fact, in its canonical factorisation $p = mq$ the monomorphism m is a strong epi, whence an isomorphism. (SA.5) becomes a particular case of the Short Five Lemma 6.1, after noting that the retractions p, p' are strong epimorphisms, hence again conormal epis.

The converse follows easily from some results proved in [3]. Let us assume (SA1–5). Now (WE.3–4) follow from (SA.3–4), after recalling that, in our hypotheses, any coequalizer p is equivalent to cok (ker p) ([3], Theorem 2.5); finally, (WE.5) holds ([3], Theorem 4.3), as well as (WE.6) ([3], 3.9).

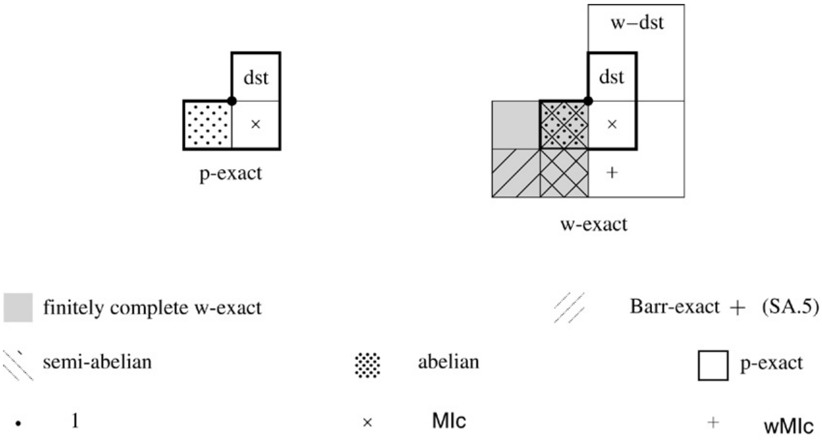
(c) Immediate from (b). □

1.6. Examples and comparisons

The categories Gp, Rng, and their categories of presheaves or sheaves on a given space are well-known to be semi-abelian (hence w-exact), as well as $(\mathsf{Set}_*)^{\mathrm{op}}$, the opposite category of pointed sets; all these are not p-exact. A characterisation of the algebraic theories whose models form a semi-abelian category can be found in [3]; for instance, any algebraic theory with a unique constant and a group operation is of this type. The category TFAb of torsion-free abelian groups is easily seen to be additive w-exact; but it is not Barr-exact (nor semi-abelian) nor p-exact (see [2], Book 2, Ch. 2).

We have already seen in Section 1.1 that $K\!-\!\mathsf{Prj}$, $\mathcal{I}$ and Mlc are p-exact, but lack products and sums. We shall construct in the next section a w-exact category wMlc which contains Mlc and plays the same role as the latter for w-exact categories: to simulate the structure of the posets of subobjects, together with their direct and inverse images; this new category is not p-exact, and again lacks products and sums.

Summarising various previous results, the settings we have considered can be represented in two diagrams. The first shows, within p-exact categories, two important sub-contexts: abelian categories and *distributive* p-exact categories, where all lattices of subobjects are so (Section 1.1). We already noticed that their intersection is reduced to trivial abelian categories (equivalent to $\mathbf{1}$), while their union does not cover p-exact categories (for instance, Mlc or $K\!-\!\mathsf{Prj}$ are neither abelian nor distributive)



The second shows that:

– w-exact categories contain two "similar" contexts: w-exact categories with finite products (or, equivalently, with finite limits) and w-distributive w-exact categories; again, their intersection is reduced to the categories equivalent to **1** (the proof is similar to the previous one, in Section 1.1), while the w-exact categories Mlc and wMlc do not belong to any of them;
– within w-exact categories with finite products, we have a chain of implications: semi-abelian $\Rightarrow$ Barr-exact with (SA.5) $\Rightarrow$ finitely complete w-exact;
– intersecting the diagram with p-exact categories (i.e., the w-exact categories where every subobject is normal), we find again the first diagram: in fact, a p-exact category with finite products amounts to an abelian category, and all terms of the previous chain reduce to this case.

Definition and Theorem 1.7 (*w-exact Functors*). *A functor $F: E \to E'$ between w-exact categories will be said to be w-exact if it satisfies these equivalent conditions:*

(a) *F preserves the zero object, monos, their pullbacks along arbitrary morphisms, conormal epis and their pushout;*
(b) *F preserves the zero object, monos and their pullbacks (of monos along monos), conormal epis and their pushout; as well as mixed pullbacks (pullbacks of monos along conormal epis, see 6.2).*
(c) *F preserves the zero object, monos, their pullbacks, kernels and conormal epis;*
(d) *F preserves monos, their pullbacks and short exact sequences.*

Proof. Plainly, (a) $\Rightarrow$ (c) $\Rightarrow$ (d). Then (d) $\Rightarrow$ (b) follows from the characterisations in terms of short exact sequences:

– of the zero object (the sequence $0 \rightarrowtail 0 \twoheadrightarrow 0$ is short exact),
– of conormal epis (they appear in short exact sequences $\bullet \rightarrowtail \bullet \twoheadrightarrow \bullet$, at the right),
– of mixed pullbacks (see 6.2) and of pushouts of conormal epis (see 6.3).

Finally (b) $\Rightarrow$ (a) follows from the factorisation property of pullbacks of monos along arbitrary morphisms (6.4). □

1.8. w-exact subcategories

If **E** is w-exact, a *w-exact subcategory* **E**$'$ is a subcategory which is w-exact in its own right and such that the inclusion is w-exact. If **E**$'$ is a full subcategory, this simply amounts to requiring that **E**$'$ be closed in **E** for

– zero object,
– pullbacks of monos (hence also kernels),
– cokernels of kernels of morphisms of **E**$'$,

of course in the obvious sense: **E**$'$ has to contain some zero object of **E**, etc.

Note that the embedding **E**$' \to$ **E** need not reflect the normality relation. For instance, in the embedding TFAb, a subgroup $A' \subset A$ of a torsion-free abelian group is a normal subobject in TFAb if and only if the abelian group A/A' is torsion free.

2. w-lattices and transfer functors

We show now that, in any w-exact category **E**, the subobjects of an object form a *w-modular w-lattice* $\mathrm{Sub}_{\mathbf{E}}(A)$, a notion introduced in [15]. Direct and inverse images of subobjects in **E** are described by a w-exact transfer functor $\mathrm{Sub}_{\mathbf{E}}\colon \mathbf{E} \to \mathsf{wMlc}$.

Definition 2.1. A *w-lattice*, or *weak lattice*, is a triple $X = (X, \leq, \triangleleft)$ such that (for all $x, y, z, t \in X$)

(wl.0) $(X, \leq)$ is a meet-semilattice with 0 and 1;
(wl.1) $\triangleleft$ is a binary relation on X (called *normality*); $0 \triangleleft 1$; $x \triangleleft x$; $x \triangleleft y \Rightarrow x \leq y$;
(wl.2) if $x \triangleleft y$ then $x \wedge z \triangleleft y \wedge z$;
(wl.3) if $x \triangleleft t$ and $y \leq t$, then $x \vee y$ exists;
(wl.4) if $x \triangleleft t$ and $y \triangleleft z \leq t$, then $x \vee y \triangleleft x \vee z$.

Note that X, generally, is not a lattice; also when it is, the only *structural* joins (used below to define homomorphisms) are the $\triangleleft$-*joins* considered in (wl.3). The following is a trivial consequence of (wl.2)

(a) if $x \triangleleft y$ and $x \leq z \leq y$, then $x \triangleleft z$.

The elements $x \triangleleft 1$ will be said to be *normal*; they form an ordered subset $\mathrm{Nrm}(X)$ which is a join-semilattice; note that the induced normality relation coincides with $\leq$, by (wl.2). We say that a w-lattice X is *normal* when every element is so, i.e. $X = \mathrm{Nrm}(X)$: then $(X, \leq)$ is a lattice and $\triangleleft$ coincides with $\leq$; thus, a lattice amounts to a normal w-lattice.

Consider now the category wLth of w-lattices and their *homomorphisms*, preserving normality, meets and $\triangleleft$-joins. A *sub-w-lattice* of a w-lattice X is a subobject in the category wLth; it can be realised as a subset Y of X with the induced order and a *finer* $\triangleleft_Y$, stable under minimum, maximum, meets and $\triangleleft_Y$-joins; it is a *regular* subobject of X if the inclusion reflects normality.

Consider also the forgetful functor $\mathsf{wLth} \to \mathsf{nPos}$, taking values in the category of $\triangleleft$-*posets* $(S, \leq, 0, 1, \triangleleft)$, i.e. sets equipped with an order relation $\leq$, minimum 0, maximum 1, and a binary relation $\triangleleft$ satisfying (wl.1), with the mappings preserving such structure. The *free w-lattice generated by a* $\triangleleft$-*poset* exists, as an easy consequence of the Freyd adjoint-functor theorem.

Indeed, in both categories, products are constructed set theoretically with componentwise structure, while equalizers are also constructed set theoretically, with induced structure. Checking the solution set condition is also classical: a morphism $f\colon S \to X$ in nPos, with values in a w-lattice X, factorises through the closure Y of its set-theoretical image under meets and existing joins of X; this subset Y, with the induced order and normality relation, is a (regular) subobject of X in wLth, whose cardinal is bounded by a cardinal determined by S: either $\aleph_0$ if S is finite, or $\#S$ otherwise.

2.2. *Weak modularity and distributivity*

A w-lattice X is said to be *w-modular*, or a *wm-lattice* if it satisfies the following axioms:

(wm.1) if $y \leq z$, $x \triangleleft t$, $y \leq t$, then $(x \vee y) \wedge z = (x \wedge z) \vee y$,
(wm.2) if $x \leq z$, $x \triangleleft t$, $y \leq t$, then $(x \vee y) \wedge z = x \vee (y \wedge z)$.

It is said to be *w-distributive*, or a *wd-lattice*, if it satisfies the stronger axioms:

(wd.1) if $x \triangleleft t$, $y \leq t$, then $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$,
(wd.2) if $x \triangleleft t$, $y \leq t$, $z \leq t$, then $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

To deduce (wm.2) from (wd.2), take $z' = z \wedge t \leq t$.

Such objects form two full subcategories of wLth, the categories wMlh, wDlh of wm- or wd-lattices and their homomorphisms. Again, we have the *free w-modular w-lattice generated by a* $\triangleleft$-*poset*. The normal w-lattices in such categories reduce to ordinary modular and distributive lattices.

2.3. $\overline{w}$-lattices

The subset $\mathrm{Nrm}(X)$ of a w-lattice X is not meet-stable, generally. For instance, consider the following wd-lattice (the trivial normality relations are understood: $0 \lhd x \lhd x$, for all elements x)

$$0 < a \wedge b < \begin{matrix} a \\ \\ b \end{matrix} < 1, \qquad 0 \lhd a \wedge b \lhd \begin{matrix} a \\ \\ b \end{matrix} \lhd 1, \tag{7}$$

where $a \wedge b$ is not normal. (Theorem 4.1 will show that this is the free wm-lattice generated by two chains, $0 \lhd a \lhd 1$ and $0 \lhd b \lhd 1$.)

We say that X is a $\overline{\mathrm{w}}$-lattice if it satisfies the following equivalent conditions

(wl.2′) if $x \lhd z$ and $y \lhd z$, then $x \wedge y \lhd z$,
(wl.2″) if $x \lhd x'$ and $y \lhd y'$, then $x \wedge y \lhd x' \wedge y'$.

More precisely, assuming (wl.0, 1), the second condition (wl.2″) is equivalent to the conjunction of (wl.2) and (wl.2′). Plainly, it implies both. Conversely, if they hold, let us assume that $x \lhd x'$ and $y \lhd y'$. By (wl.2) we have that $x \wedge y' \lhd x' \wedge y'$ and $x' \wedge y \lhd x' \wedge y'$; then, by (wl.2′), $x \wedge y \lhd x' \wedge y'$.

Analogously we consider $\overline{\mathrm{w}}$m-lattices and $\overline{\mathrm{w}}$d-lattices. If X is a $\overline{\mathrm{w}}$- (resp. $\overline{\mathrm{w}}$m-, $\overline{\mathrm{w}}$d-) lattice then $\mathrm{Nrm}(X)$ is a lattice (resp. a modular, distributive one).

2.4. wm-connections

We need now to construct a category wMlc whose objects are the w-modular w-lattices, while the morphisms, called wm-connections, are certain “covariant Galois connections” simulating direct and inverse images of subobjects in w-exact categories (and are *not* homomorphisms of w-lattices).

A *wm-connection* $u: X \to Y$ is a pair $u = (u_\bullet, u^\bullet)$, where (for all $x \in X$ and $y \in Y$)

(wmc.1) the mappings $u_\bullet: X \to Y$, $u^\bullet: Y \to X$ preserve $\le$ and $\lhd$, and satisfy $u_\bullet 0 = 0$, $u^\bullet 1 = 1$,
(wmc.2) $u^\bullet u_\bullet(x) = x \vee u^\bullet(0) \ge x$,
(wmc.3) $u_\bullet u^\bullet(y) = y \wedge u_\bullet(1) \le y$.

In particular $u_\bullet \dashv u^\bullet$ (adjoint increasing mappings between posets), so that the triangle equations hold ($u_\bullet = u_\bullet u^\bullet u_\bullet$, $u^\bullet = u^\bullet u_\bullet u^\bullet$), and each of these mappings determines the other; $u_\bullet$ preserves the existing joins while $u^\bullet$ preserves meets. wMlc is concrete and coconcrete, with the obvious forgetful functors with values in Set and $\mathsf{Set}^{\mathrm{op}}$ (sending the arrow $(u_\bullet, u^\bullet)$ to the mapping $u_\bullet$ or $u^\bullet$, respectively).

Note that the last two properties in (wmc.1) are obviously a consequence of the adjunction $u_\bullet \dashv u^\bullet$ (but we prefer to state them explicitly to avoid doubts on the meaning of (wmc.2), where the existence of $x \vee u^\bullet(0)$ is ensured by $u^\bullet(0) \lhd u^\bullet(1) = 1$). Note also that the isomorphisms of wMlc are pairs $(u_\bullet, u^\bullet)$ of inverse isomorphisms of wMlh, and can be identified with the latter.

Equivalently, one can replace (wmc.2) and (wmc.3) with

(wmc.2′) $u^\bullet(u_\bullet x \vee y) = x \vee u^\bullet y$, for every $x \in X$ and $y \lhd 1$ in Y,
(wmc.3′) $u_\bullet(u^\bullet y \wedge x) = y \wedge u_\bullet x$, for every $x \in X$ and $y \in Y$.

To deduce (wmc.2′) from (wmc.2–3), we also use the w-modularity properties

$$\begin{aligned} u^\bullet(u_\bullet x \vee y) &= u^\bullet u_\bullet u^\bullet(u_\bullet x \vee y) = u^\bullet((u_\bullet x \vee y) \wedge u_\bullet 1) = u^\bullet(u_\bullet x \vee (y \wedge u_\bullet 1)) \\ &= u^\bullet(u_\bullet x \vee u_\bullet u^\bullet(y)) = u^\bullet u_\bullet(x \vee u^\bullet(y)) = x \vee u^\bullet(y) \vee u^\bullet(0) = x \vee u^\bullet(y). \end{aligned}$$

Proposition 2.5. (a) *The category* wMlc *of wm-lattices and wm-connections has a zero-object, the one-point lattice* $0 = \{*\}$*; zero morphisms are given by*

$$0_{XY}: X \to Y, \qquad x \mapsto 0_Y, \;\; y \mapsto 1_X. \tag{8}$$

(b) *For a wm-lattice X, any element $a \in X$ determines a subobject of X in* wMlc

$$\begin{aligned} &\downarrow a = \{x' \in X \mid x' \leq a\}, \qquad m\colon \downarrow a \to X, \\ &m_\bullet(x') = x', \qquad m^\bullet(x) = x \wedge a \qquad (x' \leq a), \end{aligned} \tag{9}$$

where the subset $\downarrow a$ is structured by restricting the order relation, meets, normality and $\lhd$-joins of X (but has its own maximum, a; generally, it is not a sub-w-lattice of X, i.e. a subobject in wMlh*).*

All subobjects of X in wMlc *are of the type described above, in bijective correspondence with the elements $a \in X$;* wMlc *is well-powered. The morphism $u = (u_\bullet, u^\bullet)\colon X \to Y$ is a monomorphism if and only if $u_\bullet$ is injective, if and only if $u^\bullet$ is surjective, if and only if $u^\bullet u_\bullet = 1_X$.*

(c) *For any morphism $u = (u_\bullet, u^\bullet)\colon X \to Y$, the kernel $k\colon \mathrm{Ker}\, u \to X$ exists and is the subobject described above, for $a = u^\bullet(0)$. The morphism u has a canonical factorisation $u = mp$, with $p = \mathrm{cok}\, k$, constructed as follows*

$$\begin{tikzcd} \downarrow u^\bullet(0) \arrow[r, tail, "k"] & X \arrow[rr, "u"] \arrow[dr, "p"'] & & Y \\ & & \uparrow u^\bullet(0) \arrow[ur, "m"'] & \end{tikzcd} \qquad \begin{aligned} &k_\bullet(x') = x',\ k^\bullet(x) = x \wedge u^\bullet(0), \\ &p_\bullet(x) = x \vee u^\bullet(0),\ p^\bullet(x'') = x'', \\ &m_\bullet(x'') = u_\bullet(x''),\ m^\bullet(y) = u^\bullet(y). \end{aligned} \tag{10}$$

(Also $\uparrow u^\bullet(0)$ is structured by restricting the structure of X, but has its own minimum, $u^\bullet(0)$).

(d) *Each short exact sequence of* wMlc *is of the following kind, up to isomorphism, where $a \lhd 1_X$*

$$\downarrow a \overset{k}{\rightarrowtail} X \overset{p}{\twoheadrightarrow} \uparrow a \tag{11}$$

$$k_\bullet(x') = x', \quad k^\bullet(x) = x \wedge a, \qquad p_\bullet(x) = x \vee a, \quad p^\bullet(x'') = x''.$$

The set of normal subobjects of X is thus in bijective correspondence with the set $\mathrm{Nrm}(X)$ of normal elements $a \lhd 1_X$ of X.

Proof. Point (a) is plain. For (b), let us take an element $a \in X$. It is straightforward to verify that, restricting to the subset $\downarrow a$ the structure of X, we have a wm-lattice (with a new maximum, a); the pair $m = (m_\bullet, m^\bullet)$ is plainly a wm-connection, and a monomorphism since $m_\bullet$ is injective.

Leaving for the moment the rest of (b), let us prove (c). First, the kernel of $u = (u_\bullet, u^\bullet)\colon X \to Y$ is indeed the subobject $\downarrow u^\bullet(0)$, with the above structure and the morphism k in (10)

$$\mathrm{Ker}\, u = \downarrow u^\bullet(0) = \{x' \in X \mid x' \leq u^\bullet(0)\} = \{x' \in X \mid u_\bullet(x') \leq 0)\}. \tag{12}$$

To verify the universal property, let us take a wm-connection $f\colon Z \to X$ such that $uf = 0$. Since, for all $z \in Z$, $u_\bullet f_\bullet(z) = 0$, we have $f_\bullet(Z) \subset \downarrow u^\bullet(0)$; calling $g_\bullet\colon Z \to \downarrow u^\bullet(0)$ and $g^\bullet\colon \downarrow u^\bullet(0) \to Z$ the restrictions of $f_\bullet$ and $f^\bullet$, we do have a wm-connection $g\colon Z \to \downarrow u^\bullet(0)$ (also because $g^\bullet(u^\bullet(0)) = f^\bullet(u^\bullet(0)) = 1$, $f^\bullet(0) = g^\bullet(0)$, $f_\bullet(1) = g_\bullet(1)$) which satisfies $kg = f$; and the only one, since $k_\bullet$ is injective.

Since $a = u^\bullet(0) \lhd 1$, we can construct the cokernel of $k\colon \mathrm{Ker}\, u \to X$ as required in (10)

$$\mathrm{Cok}\ k = \uparrow u^\bullet(0) = \{x'' \in X \mid x'' \geq a\}, \qquad p\colon X \to \mathrm{Cok}\ k. \tag{13}$$

Again, it is straightforward to verify that, restricting to a subset $\uparrow a$ of X (where $a \lhd 1$) the order relation, meets, normality and $\lhd$-joins of X, we have a wm-lattice (with a *new* minimum, a). The pair $p = (p_\bullet, p^\bullet)$ is plainly a wm-connection (use (wl.4) to show that $p_\bullet$ preserves normality). Finally, for the universal property, take a wm-connection $f\colon X \to Z$ such that $fk = 0$, i.e. $f_\bullet u^\bullet(0) = 0$ or equivalently $f^\bullet(Z) \subset \uparrow u^\bullet(0)$; calling $g_\bullet\colon \uparrow u^\bullet(0) \to Z$ and $g^\bullet\colon Z \to \uparrow u^\bullet(0)$ the restrictions of $f_\bullet$ and $f^\bullet$, we do have a wm-connection $g\colon Z \to \uparrow u^\bullet(0)$ (also because $g_\bullet(u^\bullet(0)) = f_\bullet(u^\bullet(0)) = 0$) such that $gp = f$; and the only one, since $p^\bullet$ is surjective.

Putting both results together, every morphism $u\colon X \to Y$ factorises as indicated above, in (10), through $p = \mathrm{cok}\,(\ker u)$, a conormal epi; moreover the second morphism m has $m^\bullet m_\bullet(x'') = x'' \vee u^\bullet(0) = x''$ (for all $x'' \in \uparrow u^\bullet(0)$), which shows that $m_\bullet$ is injective and m is mono.

Now, coming back to the second part of (b), note that all monomorphisms are as in (10), up to isomorphism (because of our factorisation); so that it is also true that, if $m\colon X' \to X$ is mono, then $m_\bullet$ is injective. From the "triangle equations" ($u_\bullet = u_\bullet u^\bullet u_\bullet$, $u^\bullet = u^\bullet u_\bullet u^\bullet$) we deduce that $m^\bullet m_\bullet = 1$ is an equivalent condition, as well as requiring $m^\bullet$ to be surjective. Now, restricting $m_\bullet$ and $m^\bullet$, the w-lattice X' is proved to be isomorphic to its "image" in X

$$m_\bullet(X) = \{x \in X \mid x \leq m_\bullet(1)\} = \downarrow m_\bullet(1),$$

when this is equipped with the restricted structure described in (b). This completes the proof, since (d) is an obvious consequence of (c). □

Theorem 2.6 (*w-exactness*). (a) *The category* wMlc *of wm-lattices and wm-connections is w-exact and well powered. Its structure is described in* 2.5.
(b) *The full subcategories* wDlc, Mlc *and* Dlc *(of w-distributive w-lattices, modular lattices and distributive lattices, respectively) are w-exact subcategories of* wMlc *(1.8); the last two are actually p-exact.*

Proof. (a) After the previous proposition, the structure of wMlc is clear and we only have to verify the axioms (WE.3–6).

(WE.3) (Pullback of a mono). Given $u: X \to Y$ and a subobject $\downarrow b$ of Y (as in 2.5(b)), it is straightforward to verify that its counterimage is $\downarrow u^\bullet(b)$ as a subobject of X

$$\begin{tikzcd} X \arrow[r, "u"] & Y \\ \downarrow u^\bullet(b) \arrow[u, tail, "m"] \arrow[r, "v"'] & \downarrow b \arrow[u, tail, "n"'] \end{tikzcd} \qquad \begin{aligned} &v_\bullet(x') = u_\bullet(x') \qquad (x' \leq u^\bullet(b)),\\ &v^\bullet(y') = u^\bullet(y') \qquad (y' \leq b). \end{aligned} \tag{14}$$

(WE.4) Now, let us replace u with a conormal epi $p: X \to \uparrow a$ (as in (11), with $a \lhd 1$) and n with a subobject $[a, b] = \{x \in X \mid a \leq x \leq b\}$ of $\uparrow a$ (with $a \leq b$; but note that, notwithstanding our notation, $[a, b]$ is not totally ordered, in general). The pullback is calculated as above, the subobject $\downarrow p^\bullet(b) = \downarrow b$ of X

$$\begin{tikzcd} X \arrow[r, two heads, "p"] & \uparrow a \\ \downarrow b \arrow[u, tail, "m"] \arrow[r, two heads, "v"'] & {[a, b]} \arrow[u, tail, "n"'] \end{tikzcd} \qquad \begin{aligned} &p_\bullet(x) = x \vee a, \qquad p^\bullet(x'') = x'',\\ &(x'' \geq a), \end{aligned} \tag{15}$$

and v is indeed a conormal epi, as characterised in (11), since $a \leq b$ and

$$v_\bullet(x') = p_\bullet(x') = x \vee a, \qquad v^\bullet(x'') = p^\bullet(x'') = x'' \qquad (x' \leq b, x'' \in [a, b]).$$

(WE.5) (Restricted Short Five Lemma) Take a commutative diagram with short exact rows

$$\begin{tikzcd} \bullet \arrow[r, tail, "k"] \arrow[d, equal] & Y \arrow[r, two heads, "q"] \arrow[d, tail, "f"'] & \bullet \arrow[d, equal] \\ \bullet \arrow[r, tail, "h"'] & X \arrow[r, two heads, "p"'] & \bullet \end{tikzcd} \tag{16}$$

Since f is mono, we can assume that $Y = \downarrow b \subset X$, with the induced structure described in 2.5(b), and $f_\bullet$ is the inclusion. By the left square, $b = f_\bullet(1_Y) \geq h_\bullet(1) = p^\bullet(0)$. Therefore $b = b \vee p^\bullet(0) = p^\bullet p_\bullet f_\bullet(1_Y)$, and the right square gives $b = p^\bullet q_\bullet(1_Y) = p^\bullet(1) = 1_X$, i.e. $Y = X$.

(WE.6) the image of a normal mono by a conormal epi is a normal mono. Indeed, the image of k by p

$$\downarrow a \xrightarrow{k} X \xrightarrow{p} \uparrow b \qquad (a \lhd 1,\ b \lhd 1), \tag{17}$$

corresponds to the element $p_\bullet(b) = a \vee b$, which is normal in 1 because so are a, b.

(b) Plainly, the induced structures on subsets $\downarrow a$, $\uparrow b$ of a w-lattice (where $b \lhd 1$) inherit the properties of w-distributivity or normality. Thus, the full subcategories wDlc, Mlc and Dlc are stable in wMlc under zero object, pullbacks of monos and cokernels of normal subobjects, which implies the thesis (1.8). □

Theorem 2.7 (*The Transfer Functor*). (a) *For every well-powered w-exact category* $\mathbf{E}$*, there is a w-exact functor, which reflects all the "properties of w-exactness" (1.7)*

$$\mathrm{Sub}_{\mathbf{E}}\colon E \to \mathsf{wMlc}, \tag{18}$$

where $\mathrm{Sub}_{\mathbf{E}}(A)$ *is the w-modular w-lattice of* $\mathbf{E}$*-subobjects of an object* A*, while for a morphism* $f\colon A \to B$*,*

$$\mathrm{Sub}_{\mathbf{E}}(f) = (f_*, f^*)\colon \mathrm{Sub}_{\mathbf{E}}(A) \to \mathrm{Sub}_{\mathbf{E}}(B), \tag{19}$$

is the wm-connection formed of direct and inverse images of subobjects along f.
(b) *In particular, taking* $\mathbf{E} = \mathsf{wMlc}$*, this functor is isomorphic to the identity functor of* wMlc.

Proof. (A) For every object A, $\mathrm{Sub}_{\mathbf{E}}(A)$ is a poset with minimum $0\colon 0 \rightarrowtail A$ and maximum $1\colon A \rightarrowtail A$. We define the relation $x \lhd y$, for $x\colon X \rightarrowtail A$ and $y\colon Y \rightarrowtail A$, to mean that x factors as $x = yh$ where $h\colon X \rightarrowtail Y$ is a normal subobject. The intersection $x \wedge y$ of subobjects exists, so that (wl.0) holds, as well as (wl.1).
(B) Now, we consider the action of $\mathrm{Sub}_{\mathbf{E}}$ on morphisms and then we will come back to the remaining properties of $\mathrm{Sub}_{\mathbf{E}}(A)$.

For every morphism $f\colon A \to B$ in $\mathbf{E}$, direct and inverse images of subobjects are defined in the obvious way, derived – respectively – from the canonical factorisation and pullback of monos

$$\begin{aligned} &f_*\colon \mathrm{Sub}_{\mathbf{E}}(A) \to \mathrm{Sub}_{\mathbf{E}}(B), &\quad &f_*(x) = \mathrm{im}\,(fx),\\ &f^*\colon \mathrm{Sub}_{\mathbf{E}}(B) \to \mathrm{Sub}_{\mathbf{E}}(A), &\quad &f^*(y) = \text{pullback of } y \text{ along } f. \end{aligned} \tag{20}$$

Plainly, these mappings preserve the order and satisfy $f_*(0) = 0$, $f^*(1) = 1$. As to the normality relation, f_* preserves it by (WE.6) and it is easy to show that this is also true of f^*: in every category with zero object and kernels, pullbacks of normal monos can be constructed by kernels. Finally, the relation $f_* f^*(y) = y \wedge f_*(1)$ holds in every category with factorisation system (P, M) stable under M-pullbacks, while $f^* f_*(x) = x \vee f^*(0)$ is proved in 6.2 when f is a conormal epi (and essentially depends on (WE.6)), while it is obvious for f a monomorphism.

It will be useful to note that, by adjunction, f_* preserves all the existing joins; moreover, if f is a monomorphism, f_* also preserves meets

$$\begin{aligned} f_*(x \wedge y) &= f_*(f^* f_*(x) \wedge f^* f_*(y)) = f_* f^*(f_*(x) \wedge f_*(y))\\ &= f_*(x) \wedge f_*(y) \wedge f_*(1) = f_*(x) \wedge f_*(y). \end{aligned} \tag{21}$$

Moreover, we want to prove that the property (wmc.3′), in 2.4, holds when $u = p\colon A \to A/X$ is a conormal epi

$$p_*(y \wedge p^* w) = (p_* y) \wedge w, \qquad \text{for } y \in \mathrm{Sub}_{\mathbf{E}}(A),\ w \in \mathrm{Sub}_{\mathbf{E}}(B). \tag{22}$$

Let $z = p^* w \in \mathrm{Sub}_{\mathbf{E}}(A)$ and form the left diagram below, where both squares are pullbacks

$$\begin{tikzcd} Y \arrow[r, tail, "y"] & A \arrow[r, two heads, "p"] & A/X \\ Y \wedge Z \arrow[u, tail] \arrow[r, tail] & Z \arrow[u, tail, "z"] \arrow[r, two heads] & Z/X \arrow[u, tail, "w"] \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, two heads] & \bullet \arrow[r, tail, "p_* y"] & A/X \\ Y \wedge Z \arrow[u, tail] \arrow[r, two heads] & \bullet \arrow[u, tail] \arrow[ur, dashed, tail, "h"] \arrow[r, tail] & A/X \arrow[u, tail, "w"] \end{tikzcd} \tag{23}$$

Then, also their pasting is so and, factorising its rows, we get the commutative diagram at the right, which again consists of two pullbacks (by Proposition 6.4). Now, $p_*(y \wedge z)$ is the image of the diagonal of the first rectangle, $Y \wedge Z \to A/X$, or equivalently of the second, which is indeed $h = (p_* y) \wedge w$.
(C) Remaining properties of $\mathrm{Sub}_{\mathbf{E}}(A)$.

(wl.2) If $x \lhd y$ then $x \wedge z \lhd y \wedge z$. Follows from (wmc.1, 3): $x \wedge z = z_* z^*(x) \lhd z_* z^*(y) = y \wedge z$.
(wl.3) If $x \lhd t$ and $y \le t$, then $x \vee y$ exists. Follows from (wmc.2): we can assume that $t = 1\colon T \to A$; let $p\colon A \to A/X$; then $p^* p_*(y) = x \vee y$.
(wl.4) If $x \lhd t$ and $y \lhd z \le t$, then $x \vee y \lhd x \vee z$. Follows from (wmc.1–2): with the same assumptions as previously, we have that $x \vee y = p^* p_*(y)$ is normal in $x \vee z = p^* p_*(z)$.

Now, to prove (wm.1)

(wm.1) if $y \le z,\ x \lhd t,\ y \le t$, then $(x \vee y) \wedge z = (x \wedge z) \vee y$, we can equivalently assume that $z \le t$ (replacing z with $z \wedge t$); now, all our elements are below t and we can also assume that $t = 1$ (since t_* preserves meets and existing joins). Finally, we have to prove that

(i) if $y \le z$, $x \lhd 1$, then $(x \vee y) \wedge z = (x \wedge z) \vee y$.

Take $p: A \to A/X$ the cokernel of $x: X \to A$, and let $f = pz$

$$\begin{aligned}(x \vee y) \wedge z &= z_* z^*(p^* p_*(z_* z^*(y))) = z_* f^* f_* z^*(y) = z_*(z^*(y) \vee f^*(0)) \\ &= z_*(z^*(y) \vee z^*(x)) = z_* z^*(y) \vee z_* z^*(x) = y \vee (x \wedge z).\end{aligned}$$

Similarly, for (wm.2), we can assume $z \le t$, and then $t = 1$, and we only have to prove that

(ii) if $x \le z$, $x \lhd 1$, then $(x \vee y) \wedge z = x \vee (y \wedge z)$.

Take again $p: A \to A/X$ and apply (22):

$$x \vee (y \wedge z) = p^* p_*(y \wedge p^*(p_* z)) = p^*(p_* y \wedge p_* z) = p^* p_* y \wedge p^* p_* z = (x \vee y) \wedge z.$$

(D) It is now easy to check that $F = \mathrm{Sub}_{\mathbf{E}}$ is w-exact (1.7) and reflects all the "properties of w-exactness". In fact, it preserves and reflects the zero object (obviously) and

- f is mono $\Leftrightarrow f^*(0) = 0 \Leftrightarrow f^* f_*$ is the identity $\Leftrightarrow \mathrm{Sub}_{\mathbf{E}}(f)$ is mono;
- f is a conormal epi $\Leftrightarrow f_*(1) = 1 \Leftrightarrow f_* f^*$ is the identity $\Leftrightarrow \mathrm{Sub}_{\mathbf{E}}(f)$ is a conormal epi;
- F preserves and reflects finite intersections of subobjects, because it takes the subobject $x: X \rightarrowtail A$ to $\{x' \in X \mid x' \le x\}$;
- F preserves and reflects kernels, because $\mathrm{Ker}\,(\mathrm{Sub}_{\mathbf{E}}(f)) = \{x' \in X \mid x' \le f^*(0)\}$.

(E) Finally, point (b) has already been proved in 2.5(b). □

2.8. *Remarks and distributivity*

(a) By the preceding theorem (2.7(b)), every wm-lattice X, being isomorphic to $\mathrm{Sub}_{\mathsf{wMlc}}(X)$, *can be realised* as the w-lattice of subobjects of some object (X itself) in a w-exact category (and actually in a fixed one, wMlc); therefore, *no other lattice-like notion can be suitable for w-exact categories*.
(b) Say that a w-exact category is *w-distributive* if all its w-lattices of subobjects are so, i.e. if its transfer functor $\mathrm{Sub}_{\mathbf{E}}: E \to \mathsf{wMlc}$ takes values in wDlc. By 2.7(b), the category wDlc itself is w-distributive.

2.9. *Intersection of normal subobjects*

If $\mathbf{E}$ is a w-exact category, we prove below that each of the following conditions implies that all the wm-lattices of subobjects $\mathrm{Sub}_{\mathbf{E}}(A)$ are $\overline{\mathrm{w}}$-lattices (2.3), and therefore all the subsets $\mathrm{NrmSub}_{\mathbf{E}}(A)$ of normal subobjects are modular lattices

(a) cokernels exist;
(b) binary products exist.

Note that these conditions are satisfied in Gp, Rng and their categories of presheaves. On the other hand, this fact shows that the w-exact categories wMlc and wDlc do not have (all) cokernels nor binary products.

Now, the proof. First, if cokernels exist, the inclusion $\mathrm{NrmSub}_{\mathbf{E}}(A) \to \mathrm{Sub}_{\mathbf{E}}(A)$ has a left adjoint retraction

$$\mathrm{Sub}_{\mathbf{E}}(A) \to \mathrm{NrmSub}_{\mathbf{E}}(A), \qquad x \mapsto \overline{x} = \ker(\mathrm{cok}\, x), \tag{24}$$

since $\overline{x}$ is obviously the smallest normal subobject of A bigger than x. It is easy to deduce that $\mathrm{NrmSub}_{\mathbf{E}}(A)$ is closed under meets in $\mathrm{Sub}_{\mathbf{E}}(A)$; this implies the property (wl.2′) which defines $\overline{\mathrm{w}}$-lattices: if $x \lhd z$ and $y \lhd z$ in $\mathrm{Sub}_{\mathbf{E}}(A)$, applying the previous result to the object Z (where $z: Z \rightarrowtail A$) we get the thesis: $x \wedge y \le z$.

Second, let $\mathbf{E}$ have binary products. As previously, it suffices to consider two normal subobjects of A, $x_i = \ker(f_i: A \to B_i)$; now, $x_1 \wedge x_2$ is the kernel of the morphism $< f_1, f_2 >: A \to B_1 \times B_2$.

3. Subquotients and induced morphisms

Subquotients in w-exact categories are briefly examined.

3.1. Subquotients

Let **E** be a w-exact category. A subquotient H/K of the object A is a quotient of a subobject (H) of A

$$A \xleftarrow{h} H \xrightarrow{p} H/K \qquad\qquad (K \lhd H \subset A). \tag{25}$$

(After constructing the category of relations of a w-exact category [9,15], this subquotient can be equivalently described as a *monorelation*, $hp^\circ\colon H/K \to A$; cf. [18,14,15].)

Our characterisation of "mixed pullbacks" (6.2) says that the subobjects $x\colon X \rightarrowtail H/K$

$$\begin{tikzcd} K \arrow[r, rightarrowtail, "k"] & H \arrow[r, twoheadrightarrow, "p"] & H/K \\ K \arrow[u, equal] \arrow[r, rightarrowtail, "k'"'] & H' \arrow[u, rightarrowtail, "y"] \arrow[r, twoheadrightarrow, "q"'] & H'/K \arrow[u, rightarrowtail, "x"] \end{tikzcd} \tag{26}$$

correspond bijectively to the subobjects $y\colon H' \rightarrowtail H$ containing K (whence $K \lhd H'$). More explicitly, this correspondence is

$$x \mapsto p^*(x), \qquad y \mapsto p_*(y) \qquad (y \geq p^*(0)), \tag{27}$$

where $p_* p^*(x) = x$ and $p^* p_*(y) = y \vee p^*(0) = y$.

3.2. Induced morphisms

Consider a morphism $f\colon A \to B$ together with a subquotient H/K of its domain A (with $K \lhd H \subset A$) and a subquotient H'/K' of its codomain B (with $K' \lhd H' \subset B$). Provided that the usual conditions are satisfied

$$f_*(H) \subset H', \qquad f_*(K) \subset K', \tag{28}$$

the morphism f can be restricted to numerators and denominators, as in the left diagram

$$\begin{tikzcd} K \arrow[r, rightarrowtail, "k"] \arrow[d, "f''"'] & H \arrow[r, rightarrowtail, "h"] \arrow[d, "f'"] & A \arrow[d, "f"] \\ K' \arrow[r, rightarrowtail, "k'"'] & H' \arrow[r, rightarrowtail, "h'"'] & B \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, rightarrowtail, "k"] \arrow[d, "f''"'] & H \arrow[r, twoheadrightarrow, "p"] \arrow[d, "f'"] & H/K \arrow[d, "g"] \\ K' \arrow[r, rightarrowtail, "k'"'] & H' \arrow[r, twoheadrightarrow, "q"'] & H'/K' \end{tikzcd} \tag{29}$$

and we have an *induced morphism* $g\colon H/K \to H'/K'$ between our subquotients, derived from the commutative diagram with short exact rows at the right, above.

Lemma 3.3 (*Induced Morphisms*). *In the previous situation (3.2), we have:*

(a) $\operatorname{Ker} g = (H \wedge f^*(K'))/K$, $\qquad g$ is mono $\Leftrightarrow H \wedge f^*(K') \subset K$,
(b) $\operatorname{Im} g = (f_*(H) \vee K')/K'$, $\qquad g$ is a conormal epi $\Leftrightarrow f_*(H) \vee K' \supset H'$.

Proof. First, $\operatorname{Ker} g$ is a subobject of H/K; by 3.1, it can be written as H_0/K where H_0 is the following subobject of A

$$\begin{aligned} H_0 &= h_* p^* \operatorname{Ker} g = h_* p^* g^*(0) = h_* f'^* q^*(0) = h_* f'^* h'^* h'_* q^*(0) \\ &= h_* f'^* h'^*(K') = h_* h^* f^*(K') = H \wedge f^*(K'). \end{aligned}$$

Similarly, $\operatorname{im} g \subset H'/K'$ can be written as H'_0/K where H'_0 is the following subobject of B

$$\begin{aligned} H'_0 &= h'_* q^*(\operatorname{im} g) = h'_* q^* g_* p_*(H) = h'_* q^* q_* f'_*(H) = h'_* q^* q_* h'^* h'_* f'_*(H) \\ &= h'_* q^* q_* h'^* f_*(H) = f_*(H) \vee K'. \quad \square \end{aligned}$$

Corollary 3.4 (*Isomorphic subquotients*). (a) *In a w-exact category* **E**, *the following conditions imply that two subquotients H/K, H'/K' of the same object A are isomorphic*

$$\begin{aligned} &K \wedge K' \lhd H \wedge H', \qquad H \wedge K' = K \wedge H', \\ &H \subset (H \wedge H') \vee K, \qquad H' \subset (H \wedge H') \wedge K'. \end{aligned} \tag{30}$$

(b) *If K, K' and $K \wedge K'$ are normal in A, these conditions reduce to*

$$H \wedge K' = K \wedge H', \qquad H \vee K' = K \vee H'. \tag{31}$$

(If **E** is p-exact, conditions (31) are known to be equivalent to requiring that the relation from H/K to H'/K' induced by the identity be an isomorphism; cf. [14].)

4. Bifiltered objects in w-exact categories

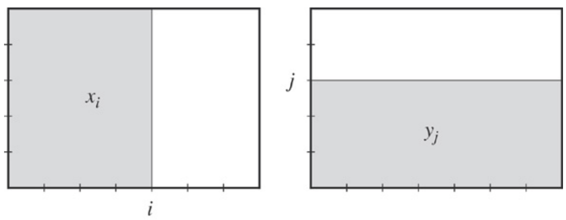
We give a "non-commutative" version of the well-known Birkhoff theorem on free modular lattices. This yields a representation of the subobjects and subquotients generated by a double filtration, as subsets of the discrete plane. The pair $(i, j) \in \mathbb{N} \times \mathbb{N}$ will be represented by the square $[i-1, i] \times [j-1,]$ of the real plane, as explained in more detail in 5.1.

Main Theorem 4.1 (*The Free wm-lattice Generated by Two Normal Chains*). *The free w-modular w-lattice $\mathbb{L} = \mathbb{L}_{m,n}$ generated by the $\lhd$-poset* (2.1) *consisting of two chains with the following normality conditions*

$$\begin{aligned} &0 = x_0 < x_1 < \cdots < x_m = 1, \qquad 0 = y_0 < y_1 < \cdots < y_n = 1, \\ &0 = x_0 \lhd x_1 \lhd \cdots \lhd x_m = 1, \qquad 0 = y_0 \lhd y_1 \lhd \cdots \lhd y_n = 1, \end{aligned} \tag{32}$$

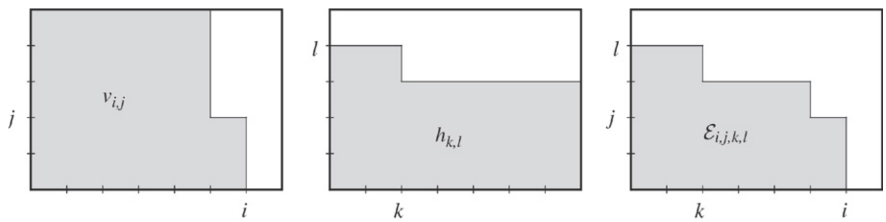
is finite and w-distributive. It can be realised as a sub-w-lattice $\mathbb{L} \subset \wp(M)$, with $M = \{1, \ldots, m\} \times \{1, \ldots, n\} \subset \mathbb{N} \times \mathbb{N}$ and

$$\begin{aligned} &x_i = \{1, \ldots, i\} \times \{1, \ldots, n\} \qquad (0 \le i \le m), \\ &y_j = \{1, \ldots, m\} \times \{1, \ldots, j\} \qquad (0 \le j \le n), \end{aligned} \tag{33}$$



Each non-null element in $\mathbb{L}$ can be written (not uniquely) as an intersection $\varepsilon_{i,j,k,l}$ of a vertical element $v_{i,j}$ and a horizontal element $h_{k,l}$, defined as follows

$$\begin{aligned} &v_{i,j} = (x_i \cap y_j) \cup x_{i-1} \ (1 \le i \le m,\ 1 \le j \le n), \\ &h_{k,l} = (x_k \cap y_l) \cup y_{l-1} \ (1 \le k \le m,\ 1 \le l \le n), \\ &\varepsilon_{i,j,k,l} = v_{i,j} \cap h_{k,l} \ (1 \le k < i \le m,\ 1 \le j < l \le n), \end{aligned} \tag{34}$$



The vertical and horizontal elements form two chains, lexicographically ordered with respect to their indices and containing the original chains (x_i), (y_j)

$$\begin{array}{ll} (v_{i,j} < v_{i',j'}) \Leftrightarrow (i < i') \text{ or } (i = i',\ j < j'), & x_i = v_{i,n}, \\ (h_{i,j} < h_{i',j'}) \Leftrightarrow (j < j') \text{ or } (j = j',\ i < i'), & y_j = h_{m,j}. \end{array} \tag{35}$$

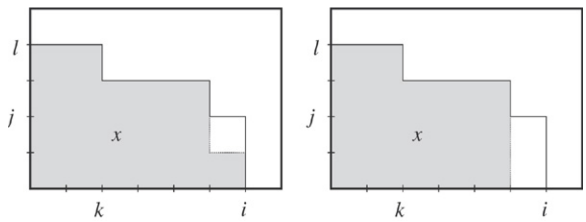
The normality relation between such elements is characterised as

$$\begin{array}{l} (v_{i,j} \lhd v_{i',j'}) \Leftrightarrow (i = i' \text{ and } j = j' - 1) \text{ or } (i = i' - 1 \text{ and } j = n), \\ (h_{i,j} \lhd h_{i',j'}) \Leftrightarrow (j = j' \text{ and } i = i' - 1) \text{ or } (j = j' - 1 \text{ and } i = m), \end{array} \tag{36}$$

while in $\mathbb{L}$ *it is characterised by the following formulas (omitting the trivial cases* $0 \lhd x \lhd x$*)*

$$\begin{array}{l} x \lhd y \ \Leftrightarrow\ \text{one can write } (x = y \cap v_{i,j},\ \text{where } v_{i,j} \lhd v_{i',j'} \text{ and } y \le v_{i',j'}) \\ \qquad\qquad \text{or } (x = y \cap h_{i,j},\ \text{where } h_{i,j} \lhd h_{i',j'} \text{ and } y \le h_{i',j'}). \end{array} \tag{37}$$

In the first case, x is obtained from y by cutting off either the upper element (i, j) of the last column of y or the last column itself, as in the following two diagrams (the second case is symmetric)



Proof. In Section 5. □

4.2. Representing subquotients

This representation of the w-lattice $\mathbb{L}$ can be extended to represent the subquotients of $\mathbb{L}$, up to isomorphism. First, let X be a wm-lattice, viewed as an object of the w-exact category wMlh. By our characterisation of subobjects and (conormal) quotients, in 2.5, a subquotient of X corresponds bijectively to a normal pair $a \lhd b$ of elements of X, which produces a pair of subobjects $\downarrow a \lhd \downarrow b$ of X in wMlh

$$\begin{array}{ll} X \xleftarrow{\ h\ } \downarrow b \xrightarrow{\ p\ } [a, b] = (\downarrow b)/(\downarrow a) & (a \lhd b \text{ in } X), \\ h_\bullet(x) = x, \quad h^\bullet(x) = x \wedge b, \quad p_\bullet(x) = x \vee a, \quad p^\bullet(x) = x. & \end{array} \tag{38}$$

Therefore, a subquotient can be identified with a *normal interval* $[a, b]$ of our wm-lattice ($a \lhd b$). Moreover, given a second subquotient $[a', b']$, Lemma 3.3 and Corollary 3.4 give sufficient conditions for their being isomorphic.

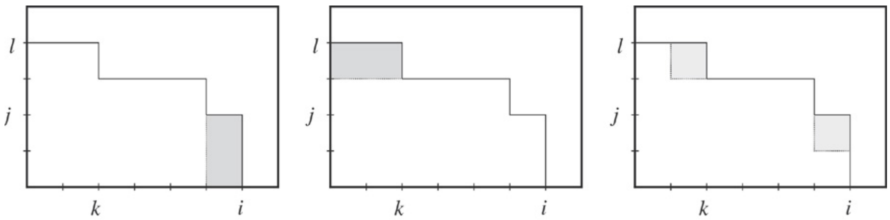
From now on, let us take for X the free wm-lattice $\mathbb{L} \subset \wp(M)$ of 4.1. In this case, there are "few" normal pairs (a, b), characterised in (37). In fact, letting $b = \varepsilon_{i,j,k,l}$, the *representative subset* $b \setminus a \subset M$ can only be empty, or b itself, or:

(i) *a vertical rectangle* $\{i\} \times \{1, \ldots, j\}$ $\quad (1 \le i \le m,\ 2 \le j \le n)$,

(ii) *a horizontal rectangle* $\{1, \ldots, k\} \times \{l\}$ $\quad (2 \le k \le m,\ 1 \le l \le n)$,

(iii) *the singleton* $\{(i, j)\}$ $\quad (1 \le i \le m,\ 1 \le j \le n)$,

(iv) *the singleton* $\{(k, l)\}$ $\quad (1 \le k \le m,\ 1 \le l \le n)$,



We prove below that the subquotient $[a, b]$ *is determined, up to isomorphism, by its representative subset* $b\backslash a \subset M$, and that a similar fact holds for every bifiltered object of a w-exact category. Note from now that our subquotient is null if and only if $a = b$, if and only if $b \setminus a = \emptyset$.

Theorem 4.3 (*Representative Subsets*). (a) *In a w-exact category* $\mathbf{E}$*, let the object* A *be equipped with two filtrations*

$$0 = X_0 \lhd X_1 \lhd \cdots \lhd X_m = 1, \qquad 0 = Y_0 \lhd Y_1 \lhd \cdots \lhd Y_n = 1. \tag{39}$$

Consider the free wm-lattice $\mathbb{L}$ *of* 4.1 *and the homomorphism* f *determined on generators by these chains*

$$f : \mathbb{L} \to \mathrm{Sub}_{\mathbf{E}}(A), \qquad f(x_i) = X_i, \qquad f(y_i) = Y_i. \tag{40}$$

Take now two normal intervals $[a, b]$, $[a', b']$ *of* $\mathbb{L}$*. Then, if* $b \setminus a = b' \setminus a'$*, the corresponding subquotients of* A *are isomorphic*

$$f(a)/f(b) \;\cong\; f(a')/f(b'). \tag{41}$$

(b) *In particular, this holds in* wMlc *itself (with* $A = \mathbb{L}$ *and* $f(x) = \downarrow x$*). Therefore, two subquotients* $[a, b]$, $[a', b']$ *of* $\mathbb{L}$ *with the same representative subset* $(b \setminus a = b' \setminus a')$ *are isomorphic w-lattices.*

Proof. We begin by proving (b). The hypothesis $b \setminus a = b' \setminus a'$ (with $a \lhd b,\ a' \lhd b'$) implies that

$$\begin{aligned} &a \wedge a' \lhd b \wedge b', \qquad b \wedge a' = a \wedge b', \\ &(b \wedge b') \vee a' \supset b', \qquad (b \wedge b') \vee a \supset b. \end{aligned} \tag{42}$$

Indeed, the last three properties are a trivial set-theoretical consequence of $b \setminus a = b' \setminus a'$ (actually equivalent to it). The first property follows rather easily from the fact that each of the normal pairs (a, b), (a', b') must fall into one of the four cases considered above, 4.2(i)–(iv). Then, applying Corollary 3.4(a) on isomorphic subquotients, $[a, b]$ and $[a', b']$ are isomorphic.

The general case (a) is an easy consequence: the wm-homomorphism f preserves the relations (42), and the conclusion follows again from Corollary 3.4(a). □

Theorem 4.4 (*Jordan–Hölder*). *In a w-exact category* $\mathbf{E}$*, let the object* A *be equipped with two filtrations*

$$0 = X_0 \lhd X_1 \lhd \cdots \lhd X_m = 1, \qquad 0 = Y_0 \lhd Y_1 \lhd \cdots \lhd Y_n = 1, \tag{43}$$

and assume that all subquotients X_i/X_{i-1} *and* Y_j/Y_{j-1} *are simple (i.e., non null and without proper quotients). Then* $m = n$ *and there is a bijection* $\varphi: [1, m] \to [1, n]$ *such that, for* $1 \le i \le m$

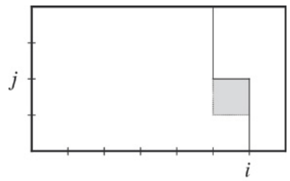
$$X_i/X_{i-1} \cong Y_{\varphi(i)}/Y_{\varphi(i)-1}. \tag{44}$$

Proof. (A) Consider the free wm-lattice $\mathbb{L}$ of 4.1 and the homomorphism determined as follows on generators

$$f : \mathbb{L} \to \mathrm{Sub}_{\mathbf{E}}(A), \qquad f(x_i) = X_i, \qquad f(y_i) = Y_i. \tag{45}$$

Let us fix an index i, $1 \le i \le m$. In the elementary column $x_i \setminus x_{i-1}$ (difference of subsets in $\wp(M)$) there is precisely one point $q_{i,j} = v_{i,j} \setminus v_{i,j-1} = \{(i, j)\}$ of the diagram whose "corresponding" subquotient $Q_{i,j}$ in $\mathbf{E}$ is not null

$$Q_{i,j} = V_{i,j}/V_{i,j-1} \neq 0, \qquad V_{i,j} = f(v_{i,j}) = (X_i \wedge Y_j) \vee X_{i-1}, \tag{46}$$



In fact, for each j ($1 \le j \le n$), consider the short exact sequence (setting $V_{i,0} = X_{i-1}$)

$$V_{i,j-1} \rightarrowtail V_{i,j} \twoheadrightarrow Q_{i,j} \tag{47}$$

if all $Q_{i,j}$ are null, one deduces that $V_{i,j-1} = V_{i,j}$ for $1 \le j \le n$, whence $X_{i-1} = V_{i,0} = V_{i,n} = X_i$, which is absurd. Let then j' be the highest index such that $Q_{i,j'} \neq 0$; from (47), again, it follows that $X_i = V_{i,n} = V_{i,j'}$; then, $Q_{i,j'} = X_i/V_{i,j'-1}$ is a quotient of X_i/X_{i-1}

$$\begin{tikzcd} X_{i-1} \arrow[r, tail] \arrow[d] & X_i \arrow[r, tail] \arrow[d, equal] & X_i/X_{i-1} \arrow[d, dashed] \\ V_{i,j'-1} \arrow[r, tail] & X_i \arrow[r, tail] & Q_{i,j'} \end{tikzcd} \tag{48}$$

and coincides with it (by simplicity); this means that $X_{i-1} = V_{i,j'-1}$, whence $V_{i,j-1} = V_{i,j-1}$ for $1 \le j < j'$; by (47), all $Q_{i,j}$ are null for $j < j'$.

Therefore, in every elementary column $x_i \setminus x_{i-1}$ of $\mathbb{L}$ there is precisely one point (i, j) such that $Q_{i,j}$ is not null (in $\mathbf{E}$), and isomorphic to X_i/X_{i-1}. By symmetry, in every elementary row $y_j \setminus y_{j-1}$ there is precisely one point $h_{i,j} \setminus h_{i-1,j} = \{(i, j)\}$ such that the subquotient $R_{i,j} = H_{i,j}/H_{i-1,j}$ in $\mathbf{E}$ is not null, and isomorphic to Y_j/Y_{j-1}.
(B) Now, for all indices i, j, we have that $Q_{i,j} \cong R_{i,j}$. This is a straightforward consequence of Theorem 4.3, because of the coincidence of their representative subsets

$$v_{i,j} \setminus v_{i,j-1} = \{(i, j)\} = h_{i,j} \setminus h_{i-1,j}. \tag{49}$$

(C) It follows that the pairs (i, j) for which $Q_{i,j} \neq 0$ coincide with the ones for which $R_{i,j} \neq 0$. Such pairs form the graph of a bijection $\varphi: \{1, \dots, m\} \to \{1, \dots, n\}$, so that $m = n$; moreover, letting $\varphi(i) = j$:

$$X_i/X_{i-1} = Q_{i,j} \cong R_{i,j} = Y_j/Y_{j-1}. \quad \square \tag{50}$$

Remarks 4.5. (a) A fortiori, Theorem 4.4 applies to a semi-abelian category $\mathbf{E}$.
(b) We have not proved that the isomorphism between $Q_{i,j}$ and $R_{i,j}$ is *canonical* (the relation induced by the identity). This will likely be a consequence of a general theory of *w-distributive* theories in w-exact categories, depending on the fact that the wm-lattice of subobjects generated by the data is w-distributive. Notice that such a theory could not be developed in the setting of semi-abelian categories, because the existence of products is in contradiction with distributivity, as recalled in the Introduction.
(c) In the last proof, we have used the representation of subquotients (4.3) in point (B). To simplify similarly point (A), we would need a deeper analysis of our representation, showing (hopefully) that it yields the universal model of the theory. This cannot be done here. And again, such developments could not be done in a setting with products.

5. Proof of the Main Theorem

This section contains the proof of Theorem 4.1.

5.1. Representing chains

Our purpose is now to give an explicit description of the free w-modular w-lattice F generated by the object of nPos consisting of two chains

$$0 = x_0 \lhd x_1 \lhd \cdots \lhd x_m = 1, \qquad 0 = y_0 \lhd y_1 \lhd \cdots \lhd y_n = 1, \tag{51}$$

where the elements 0, 1, x_i $(0 < i < m)$ and y_j $(0 < j < n)$ are pairwise distinct; the existence of F has already been ensured (2.1, 2.2).

For this we consider the set

$$M = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$$

and the distributive lattice $\wp(M)$ of subsets of M. This lattice $\wp(M)$ is certainly a w-modular w-lattice if we define the normality relation to be the inclusion of subsets. We consider in $\wp(M)$ two strictly increasing chains of elements

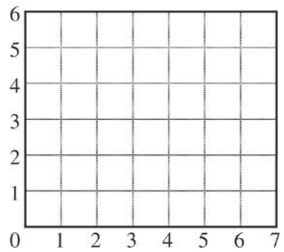
$$x_i = \{(k, j) \in M \mid k \leq i\}, \qquad y_j = \{(i, k) \in M \mid k \leq j\},$$

for all indices $0 \leq i \leq m$ and $0 \leq j \leq n$. We shall prove that the free w-modular w-lattice F is isomorphic to the sub-w-lattice E of $\wp(M)$ generated by these two chains. Let us recall that a subobject in wMlh is generally not provided with the induced structure (2.1): thus, the normality relation of E will be finer than that induced by the normality relation of $\wp(M)$.

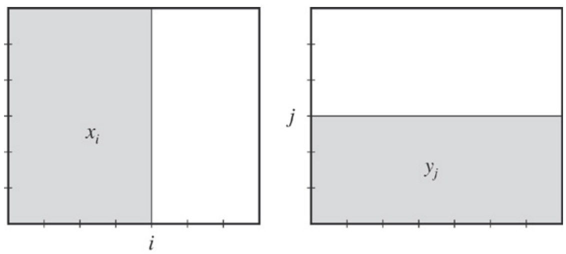
Since drawing a picture in $\mathbb{R}^2$ is easier and more intuitive than drawing a picture in $\mathbb{N}^2$, we shall represent the element $(i, j) \in M$ by the square

$$]i-1, j] \times]j-1, j] \subset \mathbb{R} \times \mathbb{R},$$

of which it is the unique point with integral coordinates. Thus we view M as the rectangle $]0, m] \times]0, n]$ in $\mathbb{R}^2$ and its elements as squares of side 1, open on the left and lower sides, closed on the right and top sides



In this representation, $1 = M$ is the full rectangle while 0 is the empty subset. The elements x_i and y_j are respectively represented as



We will write

$$\varphi: F \to \wp(M), \qquad \varphi(x_i) = x_i, \qquad \varphi(y_j) = y_j, \tag{52}$$

for the unique morphism of w-modular w-lattices fixing the generators, given by the universal property of the free object F.

5.2. Elementary elements

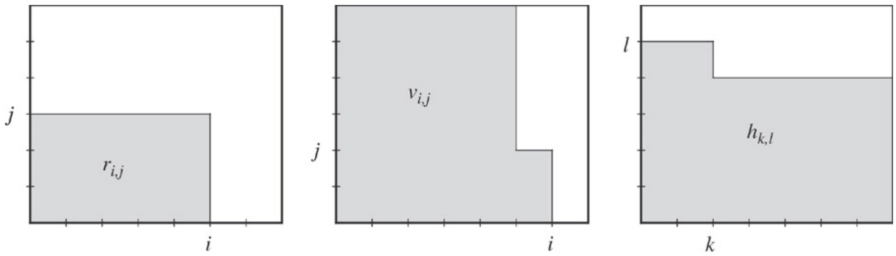
Let L be a w-modular w-lattice containing two chains, (x_i) and (y_j), with normality relations as above, in (51). These produce "elementary elements" of L, which, in the case of $\wp(M)$, will turn out to recapture the free object F. We will define them in two steps.

(a) First, the following elements exist in L, for all indices $1 \leq i \leq m$, $1 \leq j \leq n$,

1. the *rectangular* element $r_{i,j} = x_i \wedge y_j$;
2. the *vertical* element $v_{i,j} = (x_i \wedge y_j) \vee x_{i-1}$;
3. the *horizontal* element $h_{k,l} = (x_k \wedge y_l) \vee y_{l-1}$;

and contain the original chains, since $v_{i,n} = x_i$ and $h_{m,j} = y_j$.

Indeed, the element $v_{i,j}$ exists because $x_{i-1} \lhd x_i$ and $x_i \wedge y_j \leq x_i$. Analogously for $h_{k,l}$. In $\wp(M)$, these elements take the forms



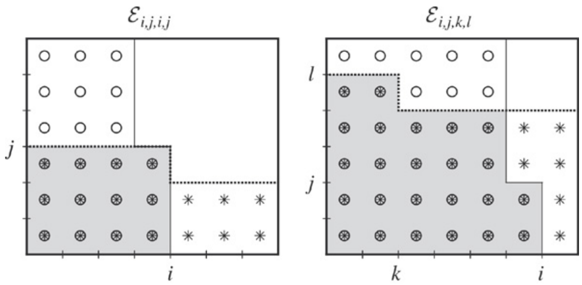
(b) Now, the *elementary elements* of L are, by definition, the following ones:

(Type 0) $\varepsilon_{0,0,0,0} = 0$;

(Type 1) $\varepsilon_{i,j,i,j} = v_{i,j} \wedge h_{i,j}$ where $1 \leq i \leq m,\ 1 \leq j \leq n$;

(Type 2) $\varepsilon_{i,j,k,l} = v_{i,j} \wedge h_{k,l}$ where $1 \leq k < i \leq m,\ 1 \leq j < l \leq n$.

We write $\mathsf{Elem}(L)$ for the poset of elementary elements of L, with the induced ordering. In $\wp(M)$, the elements of types 1 or 2 take the forms



(c) Notice that the rectangular, vertical and horizontal elements of point (a) *are all elementary*, as follows by routine computation

$\varepsilon_{i,j,i,j} = x_i \wedge y_j \qquad (1 \leq i \leq m,\ 1 \leq j \leq n)$;

$\varepsilon_{i,j,i-1,n} = v_{i,j} \qquad (1 < i \leq m,\ 1 \leq j < n)$;

$\varepsilon_{1,j,1,j} = v_{1,j} \qquad (1 \leq j \leq n)$;

$\varepsilon_{i,n,i,n} = v_{i,n} \qquad (1 \leq i \leq m)$;

$\varepsilon_{m,l-1,k,l} = h_{k,l} \qquad (1 \leq k < n,\ 1 < l \leq m)$;

$\varepsilon_{i,1,i,1} = h_{i,1} \qquad (1 \leq i \leq m)$;

$\varepsilon_{m,l,m,l} = h_{m,l} \qquad (1 \leq l \leq n)$.

The following proposition is crucial to reach our final goal.

Proposition 5.3. *The morphism $\varphi\colon F \to \wp(M)$ induces a bijective morphism of posets with top and bottom element*

$$\varphi\colon \mathsf{Elem}(F) \to \mathsf{Elem}(\wp(M)),$$

between the corresponding posets of elementary elements.

Proof. It is straightforward to observe the following property in $\wp(M)$:

$$\varepsilon_{i,j,k,l} = \varepsilon_{i',j',k',l'} \Leftrightarrow i = i',\, j = j',\, k = k',\, l = l'.$$

From the definition in terms of indices, it follows at once that $\varphi(\varepsilon_{i,j,k,l}) = \varepsilon_{i,j,k,l}$. Therefore the above property in $\wp(M)$ implies the same property in F. This proves at once the expected bijection. Notice that $0 = \varepsilon_{0,0,0,0}$, $1 = \varepsilon_{m,n,m,n}$, which are thus preserved by φ. □

5.4. Extended elementary elements

The conditions on indices in the definition of elementary elements (5.2(b)) were imposed to select a "normal form" for every elementary element and prove Proposition 5.3. We shall now get rid of the heaviness of these restrictions, which are no longer needed, and consider the "extended" elementary elements $\varepsilon_{i,j,k,l}$, under *natural* conditions of existence

$$\varepsilon_{i,j,k,l} = v_{i,j} \wedge h_{k,l}, \qquad i, k \in \{1, \ldots, m\},\;\; j, l \in \{1, \ldots, n\}. \tag{53}$$

In fact, we are only extending indices, and all such elements are elementary in the original sense (5.2(b)). It suffices to consider the cases excluded in 5.2(b), namely, $i \le k$ (written down below) and $l \le j$ (analogous):

- if $i \le k$ and $l \le j$, then $\varepsilon_{i,j,k,l} = x_i \wedge y_l$ which is elementary by 5.2(c);
- if $i = 1 \le k$ and $l > j$, then $\varepsilon_{1,j,k,l} = x_1 \wedge y_j$ which is elementary by 5.2(c), again;
- if $1 < i \le k$ and $l > j$, then $\varepsilon_{i,j,k,l} = \varepsilon_{i,j,i-1,l}$ which is elementary, since $1 \le i - 1 < i \le m$ and $1 \le j < l \le n$.

In particular, we have proved that the *set* $\mathsf{Elem}(\wp(M))$ of elementary elements of $\wp(M)$ coincides with the *set* $\mathbb{L}$ of our Main Theorem.

Remarks 5.5. Let us come back to the bijection $\varphi\colon \mathsf{Elem}(F) \to \mathsf{Elem}(\wp(M))$ (Proposition 5.3).
(a) From 5.2(c), it follows that φ induces a bijection between the rectangular (resp. the vertical, the horizontal) elements of our sets.
(b) Of course, Proposition 5.3 fails if one replaces $\wp(M)$ by an arbitrary w-modular w-lattice with two arbitrary chains. Therefore, from now on, we shall restrict our attention to the two cases F and $\wp(M)$ and their connecting morphism φ.

Proposition 5.6 (*Lexicographical Orderings*). *Let L be F or $\wp(M)$.*

1. *The vertical elements $v_{i,j} \in L$ constitute a totally ordered set, where the ordering is given by the lexicographical ordering of the indices:*

$$(v_{i,j} \le v_{i',j'}) \Leftrightarrow (i < i') \text{ or } (i = i',\; j \le j').$$

2. *The horizontal elements $h_{k,l} \in L$ constitute a totally ordered set, where the ordering is given by the (twisted) lexicographical ordering of the indices:*

$$(h_{k,l} \le h_{k',l'}) \Leftrightarrow (l < l') \text{ or } (l = l',\; k \le k').$$

Proof. The implications from right to left are obvious. The converse implications are obvious as well in $\wp(M)$. This implies the expected equivalences in F. Indeed if $v_{i,j} \le v_{i',j'}$ in F, the same relation holds in $\wp(M)$ since φ preserves the ordering; this forces the announced property of indices since the equivalences hold in $\wp(M)$. An analogous proof holds in the "horizontal" case. □

Proposition 5.7. *The posets $\mathsf{Elem}(F)$ and $\mathsf{Elem}(\wp(M))$ are meet-semilattices with top and bottom element and*

$$\varphi\colon \mathsf{Elem}(F) \to \mathsf{Elem}(\wp(M)),$$

is an isomorphism for that structure.

Proof. By Proposition 5.6 on lexicographical orderings, the meet of two vertical elements is the smallest of them and analogously for horizontal elements. The intersection of two "extended" elementary elements, as in 5.4, is thus again one of these elements, proving that $\mathsf{Elem}(F)$ and $\mathsf{Elem}(\wp(M))$ are meet semilattices. One concludes by 5.3, since a bijective morphism of meet-semilattices is an isomorphism. □

5.8. Normality conditions

We are now going to introduce a normality relation on $\mathsf{Elem}(F)$ and $\mathsf{Elem}(\wp(M))$, as the normality relation induced by the requirements $x_i \lhd x_{i+1}$, $y_j \lhd y_{j+1}$. Observe at once that in an arbitrary w-modular w-lattice L

$$\begin{aligned} x_{i-1} \lhd x_i \text{ and } y_{i-1} \lhd y_i &\Rightarrow x_i \wedge y_{j-1} \lhd x_i \wedge y_j \leq x_i \text{ and } x_{i-1} \lhd x_i \\ &\Rightarrow (x_i \wedge y_{j-1}) \vee x_{i-1} \lhd (x_i \wedge y_j) \vee x_{i-1} \\ &\Rightarrow v_{i,j-1} \lhd v_{i,j}. \end{aligned}$$

On the other hand $v_{i-1,n} = x_{i-1}$ (see 5.2(a)), thus

$$\begin{aligned} x_{i-1} \lhd x_i \text{ and } 0 \lhd x_i \wedge y_j &\Rightarrow 0 \vee x_{i-1} \lhd (x_i \wedge y_j) \vee x_{i-1} \\ &\Rightarrow v_{i-1,n} \lhd v_{i,j}. \end{aligned}$$

Analogous observations hold for horizontal elements. These necessary conditions suggest the following definition, where the two relations here above are recaptured by putting $y = v_{i,j}$ in (v1) and (v2).

Definition 5.9 (*Elementary Normality*). Let L be F or $\wp(M)$. We define the normality relation $x \blacktriangleleft y$ between two elementary elements x, y by the validity of (at least) one of the following conditions:

$$\begin{array}{lllll} \textbf{(t0)}: & x = 0; & & & \\ \textbf{(t1)}: & x = y; & & & \\ \textbf{(v1)}: & x = y \wedge v_{i,j-1}, & y \leq v_{i,j}, & 1 \leq i \leq m, & 1 < j \leq n; \\ \textbf{(v2)}: & x = y \wedge v_{i-1,n}, & y \leq v_{i,j}, & 1 < i \leq m, & 1 \leq j \leq n; \\ \textbf{(h1)}: & x = y \wedge h_{k-1,l}, & y \leq h_{k,l}, & 1 < k \leq m, & 0 \leq l \leq n; \\ \textbf{(h2)}: & x = y \wedge h_{m,l-1}, & y \leq h_{k,l}, & 1 \leq k \leq m, & 1 < l \leq n. \end{array}$$

Proposition 5.10 (*On Elementary Normality*). *Let L be F or $\wp(M)$.*

1. *The meet-semilattice $\mathsf{Elem}(L)$, provided with the normality relation $x \blacktriangleleft y$ of Definition 5.9, satisfies axioms* (wl.1) *and* (wl.2).
2. *The inclusion $\mathsf{Elem}(L) \rightarrowtail L$ preserves the normality relation, that is,*

$$\forall x, y \in \mathsf{Elem}(L), \quad x \blacktriangleleft y \Rightarrow x \lhd y.$$

3. *The isomorphism of meet-semilattices with top and bottom element*

$$\varphi\colon \mathsf{Elem}(F) \to \mathsf{Elem}(\wp(M))$$

preserves and reflects the normality relation $x \blacktriangleleft y$.

Proof. Consider the situation $x \blacktriangleleft y$ as in (v1). The conditions on indices imply necessarily $v_{i,j-1} \lhd v_{i,j}$, as observed just before Definition 5.9. Therefore

$$x = y \wedge v_{i,j-1} \lhd y \wedge v_{i,j} = y,$$

and in particular, $x \leq y$. Next given $z \in \mathsf{Elem}(L)$,

$$x \wedge z = (y \wedge z) \wedge v_{i,j-1} \lhd (y \wedge z) \wedge v_{i,j},$$

with of course $y \wedge z \leq y \leq v_{i,j}$. Thus condition (v1) applied to $y \wedge z$ yields $x \wedge z \blacktriangleleft y \wedge z$. The other cases are analogous.

The morphism φ preserves and reflects the normality relation $x \blacktriangleleft y$ simply because this relation is defined in terms of indices on the elements $v_{i,j}$ and $v_{k,l}$ (see 5.2(c), 5.3). $\square$

5.11. Describing elementary normality

Before continuing, it is useful to investigate how to draw concretely the normality relation $x \blacktriangleleft y$ in the case of $\wp(M)$. The cases $x = 0$ and $x = y$ are obvious, thus we focus on the case $0 \neq x \neq y$.

- If $x = v_{i,j-1} \wedge y$ and $y \leq v_{i,j}$ $(1 \leq i \leq m, 1 < j \leq n)$,
  x is obtained from y by cutting off the upper element (i, j) of the last column of y.
- If $x = v_{i-1,n} \wedge y$ and $y \leq v_{i,j}$ $(1 < i \leq m, 1 \leq j \leq n)$,
  x is obtained from y by cutting off the last column of y, of index i.
- If $x = h_{k-1,l} \wedge y$ and $y \leq h_{k,l}$ $(1 < k \leq m, 1 \leq l \leq n)$,
  x is obtained from y by cutting off the last element (k, l) of the upper row of y.
- If $x = h_{m,l-1} \wedge y$ and $y \leq h_{k,l}$ $(1 \leq k \leq m, 1 < l \leq n)$,
  x is obtained from y by cutting off the upper row of y, of index l.

Observe that in the case of two horizontal or two vertical elements, the normality relation $x \blacktriangleleft y$ is obvious or reduces to the cases considered before Definition 5.9.

It remains to investigate the behavior of joins. First, we check axiom (wl.3) in $\mathsf{Elem}(F)$ and $\mathsf{Elem}(\wp(M))$.

Proposition 5.12 (*Elementary Joins*). *Let L be F or $\wp(M)$. Given elementary elements $x, y, t \in L$ such that $x \blacktriangleleft y$ and $y \leq t$, the join $x \vee y$ in L is elementary and $x \blacktriangleleft x \vee y$.*

Proof. First of all, observe that it suffices to prove the result in $\wp(M)$. Indeed if this is done, consider the situation of the statement in F, which forces $x \lhd t$ by Proposition 5.10. Thus $x \vee y$ exists in F by axiom (wl.3) and considering $\varphi\colon F \to \wp(M)$, which is a morphism in wMlh, we find $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$. The result in $\wp(M)$ indicates that this last element is elementary. Proposition 5.10 forces then the result in F.

Let us now prove the result in $\wp(M)$. The cases $x = 0$, $x = t$ and $y \leq x$ are obvious. Let us assume that we are not in these cases. For $x \blacktriangleleft t$, there are four cases to consider, as observed after Proposition 5.10. We consider the two "vertical" cases; the "horizontal" ones are analogous.

The first case is when x is obtained from t by cutting off the upper element (i, j) of the last column of t. Since $y \not\leq x$, one has $(i, j) \in y$. Therefore $x \vee y = t$, which is elementary, and $x \blacktriangleleft t = x \vee y$, by assumption.

The second case is when x is obtained from t by cutting off the last column of t, of index i. Since $y \not\leq x$, y contains at least one element in the i-th column. Let us write (i, j) for the upper element of the i-th column of y. We have thus

$$x = t \wedge v_{i-1,n}, \qquad x \vee y = t \wedge v_{i,j}.$$

By 5.2(c) and 5.7, $x \vee y$ is elementary. By Proposition 5.7 again, $x \blacktriangleleft x \vee y$ because $v_{i-1,n} \blacktriangleleft v_{i,j}$. $\square$

Proposition 5.13 (*A Preliminary Isomorphism*). *Let L be F or $\wp(M)$. The elementary elements of L, with the induced ordering and the normality relation of Definition 5.9, constitute a w-distributive w-lattice, subobject of L in* wMlh. *Moreover the morphism*

$$\varphi\colon \mathsf{Elem}(F) \to \mathsf{Elem}(\wp(M)),$$

is an isomorphism of w-distributive w-lattices.

Proof. It remains to prove the validity of axioms (wl.4), (wd.1) and (wd.2). Once more by Proposition 5.10, it suffices to prove the result in $\wp(M)$. But since $\wp(M)$ is a distributive lattice, axioms (wd.1) and (wd.2) are valid in it. Thus it remains to check axiom (wl.4) in $\wp(M)$.

We must prove that given $x, y, z, t \in \mathsf{Elem}(\wp(M))$,

$$(x \blacktriangleleft t \text{ and } y \blacktriangleleft z \leq t) \Rightarrow (x \vee y \blacktriangleleft x \vee z).$$

The cases $x = 0$ and $y = z$ are obvious, while the case $y = 0$ is that of Proposition 5.12. Let us assume that we are not in these cases. The relation $x \blacktriangleleft t$ reduces again to the four cases discussed after Proposition 5.10. We treat the "vertical" cases; the "horizontal" ones are analogous.

In the first case, x is obtained from t by cutting off the upper element (i, j) of the last column of t. If $(i, j) \in y$, then $x \vee y = t$; otherwise, $y \leq x$ and thus $x \vee y = x$. The same observation holds for z instead of y. Thus the result to prove reduces, according to the case, to one of the three relations $x \blacktriangleleft x$, $x \blacktriangleleft t$, $t \blacktriangleleft t$, which hold by Proposition 5.10 or by assumption.

In the second case, x is obtained from t by cutting off the last column of t, of index i. Thus x does not have any element in the i-th column. If $y \leq t$ does not have any element in the i-th column either, then $y \leq x$ and $x \vee y = x$; the result follows then from Proposition 5.12.

Now if y has elements in the i-th column, so does z because $y \leq z$. If these elements are the same, then $x \vee y = x \vee z$ because x contains all the elements of t not in the i-th column. The conclusion is then obvious. It remains thus to consider the case where z has strictly more elements than y in the i-th column.

Since $y \blacktriangleleft z$, the considerations following Proposition 5.10 indicate again that y has been obtained from z by cutting off some element(s). But we are in the case where at least one, but not all, the elements of the i-th column of z have been cut off. In view of the four cases emphasized after Proposition 5.10, this situation splits again in two cases: to get y from z, only the upper element of the last column of z has been cut off, or z is rectangular and the whole upper row of z has been cut off.

If only the upper element (i, j) in the i-th column of z has been cut off, notice that $(i, j) \notin x$ because x does not have any element in the i-th column. Then $x \vee y$ is obtained from $x \vee z$ by cutting off this same upper element (i, j) of the last column. Therefore $x \vee y \blacktriangleleft x \vee z$, by the considerations following Proposition 5.10.

Finally if the whole upper row of index j of z has been cut off, performing the join with x leaves (i, j) as the only difference between $x \vee y$ and $x \vee z$, because x contains all the elements of t which are not in the i-th column. One concludes again that $x \vee y \blacktriangleleft x \vee z$. $\square$

Theorem 5.14 (*The Isomorphism*). *The free w-modular w-lattice F on the two given chains is isomorphic to the w-distributive w-lattice $\mathsf{Elem}(\wp(M))$ of elementary elements of $\wp(M)$. In the latter, the normality relations $x \lhd y$ and $x \blacktriangleleft y$ coincide.*

Proof. First, F coincides with its subobject $\mathsf{Elem}(F)$ of elementary elements. Indeed, the inclusion $i\colon \mathsf{Elem}(F) \rightarrowtail F$ is a monomorphism in wMlh. By universality of F, there exists a unique morphism $\psi\colon F \to \mathsf{Elem}(F)$ in wMlh which fixes the generators x_i and y_j. Therefore $i \circ \psi\colon F \to F$ is a morphism in wMlh which again fixes the generators x_i and y_j. By universality of F, $i \circ \psi$ is the identity on F, proving that the inclusion i is the identity. By Proposition 5.13, F is isomorphic to $\mathsf{Elem}(\wp(M))$. But we have also seen that in $F = \mathsf{Elem}(F)$, both normality relations coincide; and the same holds in $\mathsf{Elem}(\wp(M))$ which is isomorphic to F. $\square$

5.15. Conclusions

The Main Theorem 4.1 is now proved. By our last result, the free object F is isomorphic to $\mathsf{Elem}(\wp(M))$, which – by 5.4 – consists precisely of the poset $\mathbb{L}$ defined in 4.1. Among its elements, by 5.6, the vertical and horizontal ones are indeed ordered lexicographically. Finally, the normality relation in $\mathbb{L} = \mathsf{Elem}(\wp(M))$ is characterised in 5.11, consistently with the statement.

6. Diagrammatic lemmas in w-exact categories

We end with a few results already used above, and essentially due to Burgin [9]. Technically, they should be inserted after 1.3. Recall that "$\twoheadrightarrow$" always denotes a conormal epi, while "$\rightarrowtail$" denotes a monomorphism.

Proposition 6.1 (*Short Five Lemma*). *In a weakly exact category (Section 1.2), the property* (WE.5) *also holds for an arbitrary morphism f (without assuming it to be a monomorphism).*

Proof. The canonical factorisation $f = mq$ yields the (commutative) left diagram

$$
\begin{tikzcd}
\bullet \arrow[r, tail, "k"] \arrow[d, equal] & \bullet \arrow[r, two heads, "p"] \arrow[d, "q"] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, tail, "n"] \arrow[d, equal] & \bullet \arrow[r, two heads, "r"] \arrow[d, tail, "m"] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, tail, "n"'] & \bullet \arrow[r, two heads, "r"'] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, tail, "h'"'] & \bullet \arrow[d, tail, "h"'] & \\
\bullet \arrow[r, tail, "k"] \arrow[d, equal] & \bullet \arrow[r, two heads, "p"] \arrow[d, "q"] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, tail, "k'"'] & \bullet \arrow[r, two heads, "p'"'] & \bullet
\end{tikzcd}
\tag{54}
$$

where n is a monomorphism and r a strong epimorphism, hence a conormal one. Also the middle row is short exact, since $\ker r = \ker(p'm) = m^*(\ker p') = m^*(k') = n$ (the left lower square of monos is – trivially – a pullback). Thus, m is iso and we only need to show that also q is so, i.e. $h = \ker q$ is null. Moving to the right diagram, h factors through $k = \ker p$ (because $ph = rqh = 0$); but then $nh' = qkh' = qh = 0$, whence $h' = 0$ and $h = 0$. □

Proposition 6.2 (*Mixed Pullbacks*). *In a w-exact category, given the commutative diagram with short exact rows*

$$\begin{tikzcd}
\bullet \arrow[r, tail, "h"] & \bullet \arrow[r, two heads, "p"] & \bullet \\
\bullet \arrow[u, dashed, tail, "i"] \arrow[r, tail, "k"'] & \bullet \arrow[u, tail, "m"'] \arrow[r, two heads, "q"'] & \bullet \arrow[u, tail, "n"']
\end{tikzcd} \tag{55}$$

(MP) the right square is a pullback if and only if there is an isomorphism i which fills-in commutatively.

More precisely, for every category satisfying (WE.1–4)*, the axiom* (WE.5) *is equivalent to (MP).*

Proof. First, let us assume that (WE.1–5) hold. If the right square in (55) is a pullback, it is easy to see that mk is a kernel of p, hence $mk = h$. Conversely, given the isomorphism i, form the commutative diagram

$$\begin{tikzcd}
\bullet \arrow[r, tail, "h"] & \bullet \arrow[r, two heads, "p"] & \bullet \\
\bullet \arrow[u, tail, "n''"] \arrow[r, tail, "h'"] & \bullet \arrow[u, tail, "n'"'] \arrow[r, two heads, "p'"] & \bullet \arrow[u, tail, "n"'] \\
\bullet \arrow[u, tail, "m''"] \arrow[r, tail, "k"'] & \bullet \arrow[u, tail, "m'"'] \arrow[r, two heads, "q"'] & \bullet \arrow[u, equal]
\end{tikzcd} \tag{56}$$

where the upper squares are pullbacks, $n'm' = m$ (universal property of the right pullback) and $n''m'' = i$ (universal property of the upper rectangle). Since i is iso, n'' is a conormal epi; thus n'' is iso and m'' too. By (WE.5), applied to the lower rectangle, m' is iso: thus the right square of (1) is a pullback. Second, if (WE.1–2) hold, (MP) trivially implies (WE.5). □

Proposition 6.3 (*Pushouts of Conormal Epis*). *In a w-exact category, given the commutative diagram with exact rows*

$$\begin{tikzcd}
\bullet \arrow[r, tail, "h'"] & \bullet \arrow[r, two heads, "p'"] & \bullet \\
\bullet \arrow[u, dashed, two heads, "q_0"] \arrow[r, tail, "h"'] & \bullet \arrow[u, two heads, "q"'] \arrow[r, two heads, "p"'] & \bullet \arrow[u, two heads, "q'"']
\end{tikzcd} \tag{57}$$

(PE) the right square is a pushout if and only if there is a conormal epi q_0 which fills-in commutatively.

Proof. First assume that our category satisfies (WE.1–4, 6). If the right square of (1) is a pushout, consider the canonical factorisation $qh = h_1 q_1$. By the universal property of pushouts it is easy to see that $\operatorname{cok} h_1 = p'$; since h_1 is normal by (WE.6), it follows that $h_1 = \ker p' = h'$. Thus q_1 satisfies our condition.

Conversely, given a conormal epi q_0 making (57) commutative, let us prove that the right square is cocartesian. Consider a commutative square $p''q = q''p$, where we may assume that p'' and q'' are conormal epis, because of (WE.2). Now $p''h' = 0$ as

$$p''h'.q_0 = p''qh = q''ph = 0, \tag{58}$$

therefore p'' factors through $\operatorname{cok} h' = p'$ and the conclusion follows. The existence of pushouts of conormal epis is a trivial consequence.

Last, if (WE.1–2) hold and these pushouts do exist, (WE.6) follows clearly from (PE). □

Proposition 6.4 (*Pullbacks and Factorisations*). *In a w-exact category, given a commutative diagram*

$$\begin{tikzcd} A \arrow[r, two heads, "p"] & B \arrow[r, tail, "m"] & C \\ X \arrow[u, tail, "x"] \arrow[r, two heads, "q"'] & Y \arrow[u, tail, "y"'] \arrow[r, tail, "n"'] & Z \arrow[u, tail, "z"'] \end{tikzcd} \tag{59}$$

if the outer rectangle is a pullback, so are both squares.

(For the left square, this is a well-known fact holding in every category, for a commutative diagram where n is a monomorphism.)

Proof. Construct $y'\colon Y' \rightarrowtail B$ by pulling back z along m, and $x'\colon X' \rightarrowtail A$ by pulling back y' along p (a mixed pullback). Then insert the morphisms $v\colon Y \to Y'$ and $u\colon X \to X'$ in the obvious way, making a commutative diagram, with $y = y'v$, $x = x'u$

$$\begin{tikzcd} A \arrow[r, two heads, "p"] & B \arrow[r, tail, "m"] & C \\ X' \arrow[u, tail, "x'"] \arrow[r, two heads] & Y' \arrow[u, tail, "y'"'] \arrow[r, tail] & Z \arrow[u, tail, "z'"'] \\ X \arrow[u, tail, "u"] \arrow[r, two heads, "q"'] & Y \arrow[u, tail, "v"'] \arrow[r, tail, "n"'] & Z \arrow[u, equal] \end{tikzcd} \tag{60}$$

Now, u is an isomorphism (because pullbacks are stable under pasting); and also v is so, by the uniqueness of the canonical factorisation. □

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